A POLARIZATION INDEX FOR OVERLAPPING GROUPS

By Chiara Gigliarano*

Università degli Studi dell'Insubria

AND

Daniel Nowak and Karl Mosler

Universität zu Köln

The well-known index of income bipolarization proposed by Wolfson (1994) requires two groups to be split according to the median income and, therefore, to be non-overlapping. The aim of this paper is to propose a new polarization index in the spirit of the Wolfson index. It allows for any possible partition of the population in two or more (also overlapping) groups. The new index maintains the simplicity and immediate comprehension of the Wolfson index, though being much more flexible. An application is then provided for German and Italian income data.

1. Introduction

In the past two decades, the measurement of polarization has attracted some attention in the literature on social indicators. Based on the seminal papers by Esteban and Ray (1994) and Wolfson (1994), two different streams of the literature have emerged, underlining different aspects of the phenomenon of income polarization. The first, originating from Esteban and Ray (1994), focuses on the rise of separated income groups: polarization increases if the groups become more homogeneous internally, more separated from each other, and more equal in size. This approach is followed, among others, by Gradín (2000), D'Ambrosio (2001), Zhang and Kanbur (2001), Anderson (2004), Duclos et al. (2004), Lasso de la Vega and Urrutia (2006), Esteban et al. (2007), and Anderson et al. (2009).

The second stream, going back to Wolfson (1994, 1997), describes the decline of the middle class, measuring how the center of income distribution has been emptied. This approach is often referred to as "bipolarization" and assumes the presence of only two groups that are divided by the median income. Authors such

as Wolfson (1997), Wang and Tsui (2000), Chakravarty and Majumder (2001), Rodríguez and Salas (2003), Chakravarty et al. (2007), Deutsch *et al.* (2007), Bossert and Schworm (2008), and Chakravarty and D'Ambrosio (2010), as well as Lasso de la Vega et al. (2010), develop and extend this alternative approach. For a review of the main proposals in income polarization measurement, see Chakravarty (2009).

Rodríguez and Salas (2003) show that similarities exist between the two approaches, since the Wolfson index can be rewritten as a difference of the Gini index between groups and the Gini index within groups; thus revealing that, also according to Wolfson's measure, polarization increases with inequality between groups and decreases with inequality within groups. However, since the commonly used Gini decompositions are two-term decompositions only in the case of non-overlapping groups, Wolfson's measure and its subsequent extensions have the shortfall of assuming only two non-overlapping groups, thus considerably reducing their applicability. In the case of pure income polarization this might not be a serious drawback, but if one is interested in what Gradín (2000) calls "socioeconomic" polarization, meaning that groups are formed by characteristics other than income—for example, race or religion—groups generally overlap and this type of measure can no longer be applied.

In general, most of the existing contributions to polarization measurement focus on income as the group-forming characteristic. In contrast, Zhang and Kanbur (2001) and Gradín (2000) derive polarization measures that allow for characteristics other than income to form groups and measure income polarization among those exogenously formed groups.

In this paper, we want to combine the original Wolfson approach on income polarization with the possibility of exogenously formed groups. We propose a new polarization index based on a recent decomposition of the Gini index into two components, proposed by Okamoto (2009), which also holds in the case of overlapping groups. Analogously to the Wolfson index, the new index is also based on the Gini index and is expressed as a difference of inequality between and within groups. However, our polarization index is more general than the Wolfson index, since it does not require the groups to be split according to the median income, nor to be non-overlapping. Rather, it allows for any possible partition of the population, also with more than two groups. Therefore, our index still maintains the simplicity and immediate comprehension of the Wolfson index, though being much more flexible.

The paper is organized as follows. We first provide a short review of the existing literature on polarization measurement in Section 2. In Section 3, we introduce a new polarization measure that is based on the decomposition of the Gini coefficient proposed by Okamoto (2009) and that can be seen as a generalization of several approaches of polarization measurement, namely the approaches of Wolfson (1994, 1997), D'Ambrosio (2001), Zhang and Kanbur (2001), Rodríguez and Salas (2003), and Silber et al. (2007). As a consequence of the extension to exogenously formed groups, we have to adapt the classical polarization postulates (increased spread and increased bipolarity) and extend them to the new setting, which is done in Section 4. Next, in Section 5, the case of two groups, overlapping and not overlapping, is investigated. Section 6 covers the general case of $k \geq 2$

possibly overlapping groups. Subsequently, in Section 7 we illustrate the new measure through an empirical application based on Italian and German data. Section 8 concludes and addresses the possible extension of our approach to a measure of polarization in many attributes.

2. A Brief Review of Income Polarization Measurement

As already stressed in the introduction, the literature on univariate income polarization has its origins in the two pioneering papers of Wolfson (1994) and Esteban and Ray (1994). Independent of each other, they claimed that polarization cannot be appropriately measured by indices of income inequality. They obtained special polarization measures, Esteban and Ray (1994) from an axiomatic point of view and Wolfson (1994) based on the phenomenon of the declining middle class. In this section, we want to give a short review on the most common univariate polarization measures and their connection to well-known measures of inequality.

Let us first introduce some notation. Let $\mathbf{x}_j = \left(x_{j1}, \ldots, x_{jn_j} \right)$ be the income vector of individuals belonging to group j, and $\bar{x}_j$ be the corresponding mean income, $j = 1, \ldots, k$. The whole population has size $n = \sum_j n_j$. Denote by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ the income vector of the whole population, ordered according to the k groups, and let $\mathbf{N} = (n_1, \ldots, n_k)$ signify the vector of group sizes. The indices considered in this paper depend on $\mathbf{x}$ and $\mathbf{N}$. For simplicity, we suppress these arguments whenever possible.

Esteban and Ray (1994) set up a system of four axioms and derive the following polarization measure:

$$(1) \qquad P^{\mathrm{ER}} = K \sum_{j=1}^{k} \sum_{l=1}^{k} \left(\frac{n_j}{n} \right)^{1+\alpha} \cdot \frac{n_l}{n} \cdot |\bar{x}_j - \bar{x}_l|,$$

where $K > 0$ is a normalizing constant and $\alpha \in [1; \alpha^*]$, $\alpha^* \approx 1.6$ is the so-called polarization sensitivity. Note that the inclusion of α makes the difference between polarization and inequality in this approach, since for $\alpha = 0$ and $K = \frac{1}{2 \cdot \bar{x} \cdot n^2}$, expression (1) reverts to the Gini index.[1] According to Esteban and Ray (1994), polarization depends on the individual's sense of identification and alienation. Individuals are assumed to feel identification with other individuals belonging to the same group, and alienation versus individuals belonging to different groups. Hence, polarization increases with alienation, measured in terms of distance between group income means ($|\bar{x}_j - \bar{x}_l|$), and it decreases with identification, which depends on the size of the group to which the individual belongs ($[n_j/n]^{1+\alpha}$).

Since P^{ER} does not incorporate any within-group heterogeneity, Esteban et al. (2007) proposed an extension of the original index that is given by

---

[1]To be precise, for $\alpha = 0$ we would obtain the Gini index for classified data. This is because Esteban and Ray (1994) argue that an individual feels perfect identification with each member of his or her own subgroup, regardless of possible income differences. Therefore, each income of a group can be replaced by the respective group mean.

(2)
$$P^{\text{EGR}} = K\left[\sum_{j=1}^{k}\sum_{l=1}^{k}\left(\frac{n_j}{n}\right)^{1+\alpha}\left(\frac{n_l}{n}\right)|\bar{x}_j - \bar{x}_l|\right] - [\beta\cdot(G - G_B)],$$

where G and G_B are the total and between-group Gini indices, and $\beta \geq 0$ is an additional parameter that reflects the weight of the correction term.[2] Esteban et al. (2007) recommend that their measure should be applied after the original vector of incomes has been grouped by a statistical approach that minimizes within-group dispersion. Gradín (2000) proposed the same correction term to the original index, though in his approach groups can be defined according to variables different from income.

However, Lasso de la Vega and Urrutia (2006) point out that these extended measures violate some basic properties of polarization. In particular, according to equation 2, greater inequality between groups and at the same time constant inequality within groups may be associated with a decline in polarization. Therefore, they propose an alternative extension of the ER index of polarization, which is given by

(3)
$$P^{\text{LU}} = K\sum_{j=1}^{k}\sum_{l=1}^{k}\left(\frac{n_j}{n}\right)^{1+\alpha}\left(\frac{n_l}{n}\right)(1 - G_j)^{\beta}|\bar{x}_j - \bar{x}_l|$$

with G_j being the Gini coefficient of group j, $\alpha \in [1, 1.6]$, $\beta \geq 0$ and $K > 0$.
The second stream of the literature on income polarization is based on the polarization measure of Wolfson (1994). Additional to the existing notation, let m denote the median of the incomes and $L(z)$ be the value of the Lorenz curve at the z-quantile of $\mathbf{x}$. Wolfson (1994) proposes a polarization curve analogous to the Lorenz curve for inequality measurement and defines his polarization measure as follows:

(4)
$$P^W = \frac{2\bar{x}}{m}\cdot(1 - 2\cdot L(0.5) - G).$$

In this formulation, the link between polarization and inequality measurement is obvious, since the Gini index is used explicitly. By using a subgroup decomposition of the Gini index, we can make this link even more explicit. Let us assume that the population is divided into two groups, where the first group, L, consists of all individuals with incomes below the median income and the second group, H, consists of all individuals with higher incomes. Let us denote the two resulting income vectors as $\mathbf{x}_L$ and $\mathbf{x}_H$, with corresponding means $\bar{x}_L$ and $\bar{x}_H$ and group shares $\frac{n_L}{n} = \frac{n_H}{n} = \frac{1}{2}$, assuming n even. Also, let x_{Li} and x_{Hi} be income of the ith individual in groups L and H, respectively.

Then, the overall Gini index G (or $G(\mathbf{x})$) can be decomposed into a within-group, G^W, and a between-group, G^B, component as follows:

(5)
$$G(\mathbf{x}) = \frac{1}{2\bar{x}n^2}\sum_{i=1}^{n}\sum_{m=1}^{n}|x_i - x_m|$$

---

[2]Esteban et al. (2007) claim that incomes should be normalized to unity, so they remove the scaling parameter K in their measure.

(6)
$$= \underbrace{\frac{\bar{x}_L}{4 \cdot \bar{x}} \cdot G(\mathbf{x}_L) + \frac{\bar{x}_H}{4 \cdot \bar{x}} \cdot G(\mathbf{x}_H)}_{G^W} + \underbrace{0.5 - L(0.5)}_{G^B}.$$

Using this decomposition, one can rewrite the Wolfson measure according to

(7)
$$P^W = \frac{2\bar{x}}{m} \cdot (G^B - G^W).$$

Therefore, the Wolfson polarization measure is a normalized function of the difference between the Gini index between groups, G^B, and the Gini index within groups, G^W.[3] As Rodríguez and Salas (2003) pointed out, this formulation can be used to distinguish between polarization and inequality and to establish a link between the two streams of polarization measurement. Clearly, there is a difference between adding up the two components, as done in inequality measurement, and taking the differences as in polarization measurement. Moreover, the between-group component can be seen as a measure of the alienation between groups and the within-group component as a measure of the identification within groups, two basic ideas in the approach of Esteban and Ray (1994).

Several other polarization measures are explicitly constructed as functions of inequality measures. In particular, Rodríguez and Salas (2003) propose an extension of Wolfson's polarization measure that includes an additional sensitivity parameter v. Their measure is based on a subgroup decomposition of the extended Gini coefficient introduced by Donaldson and Weymark (1980) in the case of population divided by the median, and is defined as follows:

$$P^{RS}(\mathbf{x},v) = G^B(\mathbf{x},v) - G^W(\mathbf{x},v), \quad v \in [2,3].$$

The idea of measuring polarization in terms of the difference of a between-group inequality component and a within-group inequality component (measured through the Gini index) is also present in the polarization index of Silber et al. (2007), which is defined as follows:

$$P^{SDH} = \frac{G^B - G^W}{G},$$

where it is assumed that the population is divided by the median income.

Rather than using the difference between the two inequality components, Zhang and Kanbur (2001) use the ratio of these two components and define their polarization measure as follows:

$$P^{ZK} = \frac{I^B}{I^W},$$

[3]This is not a new result (see, e.g., Rodríguez and Salas, 2003). We discuss it here since we will compare this Gini decomposition with the one used for our new polarization index. Note that Rodríguez and Salas (2003) obtained the Wolfson expression as in equation 7 from a "geometric" reasoning based on the Lorenz curve.

where I^B and I^W are the between- and within-group components of Theil's measure of inequality I.[4] Since this measure is not defined if the within-group inequality is zero, Silber *et al.* (2007) propose the following modification:

$$P^{ZK^*} = \frac{P^{ZK} - 1}{P^{ZK} + 1} = \frac{I^B - I^W}{I}.$$

3. A New Polarization Index Based on Gini Decomposition

In this section, we will construct a polarization measure that is in line both with the intuition of Wolfson bipolarization and with the ideas of identification and alienation of Esteban and Ray (1994). Therefore, in the following paragraphs we will introduce some basic requirements that our polarization index has to meet.

First, polarization is found to increase with respect to identification within the groups. Generally, it is assumed that identification within a group is reflected by its inequality, in the sense that higher inequality within groups means lower identification and vice versa.

Second, polarization is seen to decrease with respect to alienation between groups. Alienation is measured in an obvious way via between-group inequality (Esteban and Ray, 1994); the higher the inequality between the groups, the higher is the alienation.

Note that this approach is also in line with the intuition of Wolfson bipolarization. If we split the population at the median, then the inequality within the two new groups is smaller than the inequality within the original population, and the inequality between the two new groups becomes larger than zero, both factors increasing polarization.

Third, a polarization measure has to account for the differences in group sizes in such a way that for a given number of groups k and given within- and between-group inequality, the more equally sized the groups are, the higher is the polarization (see, amongst others, D'Ambrosio, 2001 Esteban and Ray, 1994 Gigliarano and Mosler, 2009). In Wolfson's approach this issue is implicit, since the groups are divided by the median, and hence are equally sized.

In summary, a measure of polarization should increase with inequality between groups, and decrease with inequality within groups. It may also increase with the degree of homogeneity among group sizes.[5]

---

[4]Zhang and Kanbur (2001) use Theil's measure instead of the Gini index. In principle, the formulation of their polarization measure allows for any decomposable inequality index that satisfies the Pigou–Dalton transfer axiom.

[5]Observe that not only the polarization measures of the Wolfson perspective fit into this concept, but also the measures based on the Esteban and Ray approach are in line with this idea. For example, in the case of two groups, the measure by Esteban et al. (2007) can be written as $P^{EGR} = G^B(\mathbf{x}) \cdot S(\mathbf{N}) - G^W(\mathbf{x})$. Here, G^B denotes the inequality between, G^W the inequality within, and $S(\mathbf{N})$ the homogeneity of the group sizes. It can be seen immediately that P^{EGR} meets the three requirements mentioned above.

With respect to the inequality measure, we will focus on the well-known Gini index of inequality, mainly for two reasons. First, out of many existing measures, the Gini index remains the most popular and commonly used measure of inequality. Second, over the decades, several alternative decompositions of the Gini index have been proposed in the literature (see Bhattacharya and Mahalanobis, 1967; Dagum, 1997; Pyatt, 1976; Yitzhaki and Lerman, 1991). A recent proposal has been formulated by Okamoto (2009), which is the decomposition that we consider for measuring polarization and that we now briefly describe.

3.1. *Okamoto's Decomposition*

Consider a vector $\mathbf{x}$ of incomes that are, w.l.o.g., increasingly ordered; that is, $x_1 \leq x_2 \leq \ldots \leq x_n$. Let us recall the Gini mean difference

$$\Delta = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{m=1}^{n} |x_i - x_m|, \tag{8}$$

which corresponds to the numerator of the Gini index G.

The Gini mean difference and the Gini index for group j are, respectively,

$$\Delta_j(\mathbf{x}_j) = \frac{1}{n_j^2} \sum_{i=1}^{n_j} \sum_{m=1}^{n_j} |x_{ji} - x_{jm}|, \quad j = 1, \ldots, k, \tag{9}$$

and

$$G_j(\mathbf{x}_j) = \frac{\Delta_j(\mathbf{x}_j)}{2\bar{x}_j} = \frac{1}{2\bar{x}_j n_j^2} \sum_{i=1}^{n_j} \sum_{m=1}^{n_j} |x_{ji} - x_{jm}|, \quad j = 1, \ldots, k. \tag{10}$$

Finally, the Gini mean difference between group j and group l is defined as follows:

$$\Delta_{jl}(\mathbf{x}_j, \mathbf{x}_l) = \frac{1}{n_j n_l} \sum_{i=1}^{n_j} \sum_{m=1}^{n_l} |x_{ji} - x_{lm}|, \quad j, l = 1, \ldots, k. \tag{11}$$

Obviously, $\Delta_{jj}(\mathbf{x}_j, \mathbf{x}_j) = \Delta_j(\mathbf{x}_j)$. Okamoto (2009) provides a new two-term-decomposition of the Gini index, given by the following:

$$G = G_{\text{Oka}}^W + G_{\text{Oka}}^B, \tag{12}$$

where the within-group component is defined as

$$G_{\text{Oka}}^W = \sum_{j=1}^{k} \frac{n_j}{n} \frac{\bar{x}_j}{\bar{x}} G_j(\mathbf{x}_j) \tag{13}$$

and the between-group component is

$$G_{\text{Oka}}^B = \frac{1}{2\bar{x}} \sum_{j=1}^{k} \sum_{l=1}^{k} \frac{n_j n_l}{n^2} cv(\mathbf{x}_j, \mathbf{x}_l), \tag{14}$$

where the term $cv(\mathbf{x}_j, \mathbf{x}_l)$ is referred to by Okamoto (2009) as the "Cramèr coefficient of variation" between groups j and l, since it recalls the two-sample Cramer – von Mises test statistic (see, e.g., Anderson, 1962) and is defined as follows:

$$(15) \quad \begin{aligned} cv(\mathbf{x}_j, \mathbf{x}_l) &= \tfrac{1}{n_j n_l} \sum_{i=1}^{n_j} \sum_{m=1}^{n_l} |x_{ji} - x_{lm}| \\ &\quad - \tfrac{1}{2n_j^2} \sum_{i=1}^{n_j} \sum_{m=1}^{n_j} |x_{ji} - x_{jm}| - \tfrac{1}{2n_l^2} \sum_{i=1}^{n_l} \sum_{m=1}^{n_l} |x_{li} - x_{lm}| \\ &= \Delta_{jl}(\mathbf{x}_j, \mathbf{x}_l) - \tfrac{\Delta_j(\mathbf{x}_j)}{2} - \tfrac{\Delta_l(\mathbf{x}_l)}{2}. \end{aligned}$$

Note that $cv(\mathbf{x}_j, \mathbf{x}_l) = 0$ if $j = l$. Alternatively, we may write equation 14 as follows:

$$(16) \quad G_{\text{Oka}}^B = \frac{1}{2\bar{x}n^2} \sum_{j,l=1, j \neq l}^{k} \sum_{i=1}^{n_j} \sum_{m=1}^{n_l} |x_{ji} - x_{lm}|.$$

In most of the known decompositions of the Gini index, the between-group inequality component is based on differences of group mean incomes. In particular, inequality between groups becomes null if and only if the group means are the same, regardless of how different the subgroup distributions might be. This can be considered as a drawback for our class of polarization measures, as it would take into account only one parameter of the distribution. Differently, in Okamoto's decomposition the inequality between groups takes into account the whole subgroup distribution rather than only the difference in the means. Therefore, when using Okamoto's decomposition, the between-group inequality is zero if and only if all group distributions are identical and concentrated at a single point.[6] This is a rather strong condition for having zero between-group inequality, but—at least in our opinion—it fits better to the concept of group alienation.

Looking at the within-group component G_{Oka}^W, Okamoto's decomposition aggregates the inequality within each group using as weight the share of the group and its relative mean income. As a consequence, the weights sum up to one, and the within-group component can be considered properly as a weighted average of the subgroup inequalities. This appears to be more natural than using the square of the group shares and their relative means as weights, as in most of the other Gini index decompositions.

It is particularly interesting for polarization measurement that Okamoto (2009) allows for a two-term decomposition of the Gini index, while almost all the other proposals are three-term decompositions. One important exception is Dagum's decomposition, which can be rewritten as the sum of the within-group Gini index and the gross between-group Gini component. However, the latter is not a minimum when the subgroup distributions or means are all identical. Moreover, differently from the three-term decompositions, Okamoto's decomposition is consistent with multilevel grouping.

---

[6]Note that the equality of group means is, therefore, not a sufficient condition for $G_{\text{Oka}}^B = 0$. Hence, Okamoto's decomposition does not satisfy the property of additive decomposability defined in Shorrocks (1980), which is, however, considered quite restrictive, as it allows the between-group component to rely only on the first moment of each subgroup distribution (see, e.g., Ebert, 2010).

Our last reason for employing Okamoto's decomposition is that it is available not only for the unidimensional Gini index but for some of its multidimensional extensions as well, such as for the Distance Gini index and for the Volume Gini index proposed in Koshevoy and Mosler (1997). Since we also want to outline the possibility of constructing multidimensional polarization indices, this is a nice property, since it allows us to handle uni- and multidimensional polarization measures in the same framework.

Note that in using Okamoto's decomposition for polarization measurement, we do not obtain the Wolfson measure as a special case. Indeed, it can be shown that the expressions (4) and (7) of the Wolfson polarization index are equivalent only if the Gini index is decomposed using the decompositions of Dagum (1997) or Bhattacharya and Mahalanobis (1967), but not Okamoto's method. However, this is not a serious drawback, since this approach will allow us to analyze situations where the population is divided into subgroups according to characteristics other than income and where group income distributions overlap. For the sake of simplicity, henceforth, G^B and G^W will denote, respectively, the components G^B_{Oka} and G^W_{Oka} of Okamoto's decomposition.

3.2. A New Polarization Index

As already mentioned, our new polarization index should decrease with respect to the within-group inequality, increase with respect to the between-group inequality, and depend on the homogeneity of the group sizes.

To meet these requirements in a simple and transparent way, we propose a new polarization index, which is defined according to

$$(17) \qquad \tilde{P} = \tilde{P}(\mathbf{x},\mathbf{N}) = S(\mathbf{N}) \cdot \left[G^B(\mathbf{x},\mathbf{N}) - G^W(\mathbf{x},\mathbf{N}) \right].$$

For G^B and G^W, we use the components of Okamoto's decomposition (equations 14 and 13, respectively) and for $S(\mathbf{N})$ we employ the inverse of Herfindahl's measure of concentration,

$$(18) \qquad S(\mathbf{N}) = \left[\sum_{j=1}^{k} \left(\frac{n_j}{n} \right)^2 \right]^{-1},$$

which equals k if all groups have the same size, and approaches 1 if all but one group tend to become empty.[7] Obviously, the polarization index given by equation 17 is not normalized; it can actually take a negative value if the inequality within groups exceeds the inequality between groups. In order to normalize $\tilde{P}(\mathbf{x},\mathbf{N})$, we normalize each of its components. Starting with the group size homogeneity component, the inverse Herfindahl index is normalized by

$$(19) \qquad \frac{1}{k-1} \left[\left[\sum_{j=1}^{k} \left(\frac{n_j}{n} \right)^2 \right]^{-1} - 1 \right].$$

---

[7]We use the factor $S(\mathbf{N})$ since the difference $G^B(\mathbf{x},\mathbf{N}) - G^W(\mathbf{x},\mathbf{N})$ does not always increase with increasing homogeneity.

To normalize the inequality difference $D = G^B - G^W$, we have to determine its minimum and maximum. Then

$$(20) \qquad \frac{D - \min D}{\max D - \min D}$$

ranges between 0 and 1. Note that the minimum value of D depends on the vector $\mathbf{N}$ of group sizes. Hence, determining the minimum value in the case of overlapping groups is not a trivial exercise. A detailed derivation of the minimum and maximum values of D for two generic, also overlapping, groups is included in Appendix A (in the Online Supporting Information). Finally, as our normalized index of polarization, we introduce the following:

$$(21) \qquad P = P(\mathbf{x}, \mathbf{N}) = \frac{1}{k-1} \left[\left[\sum_{j=1}^{k} \left(\frac{n_j}{n} \right)^2 \right]^{-1} - 1 \right] \cdot \frac{D - \min D}{\max D - \min D}.$$

Note that in this analysis we do not intend to directly compare scenarios characterized by a different number of groups; for example, *ceteris paribus* our measure does not assign a higher level of polarization in the case of two equally sized groups than in the case of three equally sized groups.[8]

In this section, we have introduced a new polarization index that is based on the difference between inequality components obtained from Okamoto's decomposition of the Gini index. In contrast to Wolfson's approach, the new polarization index can be computed for any number of groups, which may also be overlapping. In the next section, we will focus on the properties of the new polarization index.

4. THE PROPERTIES OF THE NEW INDEX

An index of polarization is a function $P: \mathbb{R}_+^n \times \mathbb{N}^k \to \mathbb{R}, (\mathbf{x}, \mathbf{N}) \mapsto P(\mathbf{x}, \mathbf{N})$. It is supposed to satisfy a number of postulates, which we divide into two groups: the first group regards general aspects of measurement, while the second group contains those postulates that are specific to polarization and distinguish a polarization measure from, for example, a measure of inequality or concentration.

The group of general postulates of measurement includes the following postulates:

P1 Continuity $\quad \mathbf{x} \mapsto P(\mathbf{x}, N)$ is continuous on $\mathbb{R}_+^n$.

P2 Within-Group Anonymity $\quad P(\mathbf{x}, N)$ is invariant to permutations of individual labels within groups. For every group j, given a permutation π_j of $\{1, 2, \dots, n_j\}$, it holds that

---

[8]If different numbers of groups are to be compared, the index may be augmented by a factor that properly decreases with k. For example, one could consider, as an alternative version of expression (19), the following:

$$S(\mathbf{N}) = \frac{\left(\sum_{j=1}^{k} \left(\frac{n_j}{n} \right)^2 \right)^{-1} - 1}{k-1} \frac{\sqrt{2}}{\sqrt{k}}.$$

But then the normalization property (see the following postulate P5) would no longer be satisfied.

$$P(\mathbf{x}_1, \dots, \mathbf{y}_j, \dots, \mathbf{x}_k, N) = P(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_k, N),$$

where $\mathbf{x}_j = (x_{j1}, x_{j2}, \dots, x_{jn_j})$ and $\mathbf{y}_j = (x_{j\pi_j(1)}, x_{j\pi_j(2)}, \dots, x_{j\pi_j(n_j)})$.

P3 Replication Invariance   For any $\ell \in \mathbb{N}$: $P(\mathbf{x}, N) = P(\underbrace{(\mathbf{x}, \dots, \mathbf{x})}_{\ell \text{ times}}, \ell N)$.

P4 Scale Invariance $P(\mathbf{x}, N)$ is invariant to any change in the unit of measurement of income: for any scalar $\lambda > 0$, $P(\mathbf{x}, N) = P(\lambda\mathbf{x}, N)$.

P5 Normalization $P(\mathbf{x}, N)$ should be normalized between 0 and 1.

For the new polarization index, we can summarize its measurement properties by means of the following proposition:

> **Proposition 1.** *The polarization index given by equation 21 satisfies the postulates P1–P5 for any number of groups k.*

Proof Obvious.

Additional to these general postulates of measurement, there are two postulates that are specific to polarization:[9]

P6 Increased Spread [IS]   Given two groups divided by the median, a regressive Pigou–Dalton transfer between two individuals belonging to different groups should increase polarization.

P7 Increased Bipolarity [IB]   Given two groups divided by the median, if inequality within at least one group reduces, by means of a progressive Pigou–Dalton transfer, polarization should increase.

However, the two polarization postulates P6 (IS) and P7 (IB) are not appropriate if the groups do overlap, since they are defined referring to two groups divided by the median. Therefore, we now introduce a modified version of IS (postulate P6* below) and of IB (postulate P7* below), which are adapted to the case of overlapping groups and are defined for the general case of $k \geq 2$ groups.

P6* Modified Increased Spread [MIS]   Let us order the k groups increasingly according to their means, so that $\bar{x}_1 \leq \bar{x}_2 \leq \dots \leq \bar{x}_k$. This property states that if (i) incomes x_{1i}, for $i = 1, \dots, n_1$, of all individuals belonging to group 1 are decreased by the same positive value $\beta > 0$ (thus becoming $y_{1i} = x_{1i} - \beta$) and, at the same time, (ii) incomes x_{km}, for $m = 1, \dots, n_k$, of all the individuals belonging to group k are increased by the same positive value $\alpha = \frac{n_1}{n_k}\beta$, so that the overall mean is not affected (thus becoming $y_{km} = x_{km} + \alpha$), then polarization should increase.

In other words, while the Increased Spread property considers a Pigou–Dalton transfer that involves only two individuals belonging to two

[9]For a detailed formulation of these propositions, see, for example, Wang and Tsui (2000).

non-overlapping groups, the Modified Increased Spread property assumes that all individuals in the group with the lowest income mean become poorer and that all individuals of the group with the highest income mean become richer. This change should reasonably increase polarization.

> **P7* Modified Increased Bipolarity [MIB]**   Let us order the k groups increasingly according to their means, so that $\bar{x}_1 \leq \bar{x}_2 \leq \ldots \leq \bar{x}_k$. If the inequality within at least one group reduces, then P should increase. In particular, if a progressive transfer occurs between two individuals g^* and h^* belonging to the same group (w.l.o.g., we assume group 1), so that income x_{1h^*} is reduced by $\varepsilon > 0$ and income x_{1g^*} is augmented by $\varepsilon > 0$, then polarization should increase.

The Modified Increased Bipolarity property is an intuitive extension of Increased Bipolarity in the case of more than two groups. It states that any progressive transfer occurring within any subgroup distribution should increase polarization.

Together with the original versions, these two modified postulates will be investigated in detail in Sections 5 and 6.

5. Polarization of Two Groups

We now discuss the polarization properties of the new index proposed in the case of $k = 2$ groups, both non-overlapping and overlapping. We investigate the postulates of Increased Spread (P6) and Increased Bipolarity (P7) as well as their extended versions of Modified Increased Spread (P6*) and Modified Increased Bipolarity (P7*) that cope with the situation of overlapping groups. We also discuss the two extreme scenarios of minimum and maximum polarization.

5.1. *Two Non-overlapping Groups*

Proposition 2 below discusses the scenarios of minimum and maximum polarization for the case of two non-overlapping groups.

> **Proposition 2** [Minimum and maximum polarization of two non-overlapping groups]
>
> *(a) The polarization index P defined in equation 21 is maximal (equal to 1) if the groups are of equal size, at maximal distance from each other, and with no within-group inequality, which means, for a given overall mean $\bar{x}$, if $x_1 = (0,\ldots,0)$ and $\mathbf{x}_2 = (2\bar{x}, \ldots, 2\bar{x})$.*
>
> *(b) The case of minimal polarization depends on group sizes. For two groups of equal size, the polarization index P defined in equation 21 is minimal and equal to 0 if all individuals have the same income, which means, for a given overall mean $\bar{x}$, if $\mathbf{x}_1 = \mathbf{x}_2 = (\bar{x}, \ldots, \bar{x})$.*

Regarding the polarization-specific postulates of Increased Spread (IS) and Increased Bipolarity (IB), the following proposition holds:

Proposition 3 [IS and IB for two non-overlapping groups]

Let $k = 2$ and assume that the two groups are divided by the median, so that they do not overlap. Then the polarization index P defined in equation 21 always satisfies the IB postulate, while it satisfies the IS postulate if the number of individuals whose income is between the incomes of the two individuals involved in the regressive transfer is smaller than $\frac{2}{3}(n-1)$.

For proof of Proposition 3 see Appendices B.1 and B.3, respectively (in the Online Supporting Information).

According to Proposition 3, in the case of two non-overlapping groups, a sufficient condition for the Increased Spread (IS) property to hold is that at most two thirds of the population has income between the incomes of the two individuals involved in the IS transfer. Roughly speaking, consider an individual h with income below the median and an individual g with income above the median, so that the proportion of individuals between them does not exceed two thirds: if the distance between the incomes of individuals h and g increases, then the proposed polarization index will increase.

5.2. *Two Overlapping Groups*

Proposition 4 below discusses the scenarios of minimum and maximum polarization for the case of two overlapping groups.

Proposition 4 [Minimum and maximum polarization of two overlapping groups]

(a) The polarization index P defined in equation 21 is maximal (equal to 1) if the groups are of equal size, at maximal distance from each other, and with no within-group inequality, which means, for a given overall mean $\bar{x}$, if $x_1 = (0,\ldots,0)$ and $\mathbf{x}_2 = (2\bar{x}, \ldots ,2\bar{x})$.

(b) The case of minimal polarization depends on group sizes. For two groups of equal size, the polarization index P defined in equation 21 is minimal and equal to 0 if distributions of the two groups perfectly overlap and the within-group inequality is maximal, which means, for a given overall mean $\bar{x}$, if $\mathbf{x}_1 = \mathbf{x}_2 = (0, \ldots ,0, \frac{n\bar{x}}{2})$.

Regarding the polarization-specific postulate of Modified Increased Spread (MIS), the following holds:

Proposition 5 [MIS for two overlapping groups]

In the case of $k = 2$ groups, the postulate of Modified Increased Spread $(P6^{*})$ is satisfied by the polarization index P defined in equation 21 if

$$\sum_{i=1}^{n_1} \left[(\alpha + \beta) \sum_{m=1}^{n_2} \left[\mathbf{1}(y_{2m} > y_{1i}) - \mathbf{1}(y_{2m} < y_{1i}) \right] + \sum_{\substack{m:(x_{2m} < x_{1i}) \\ \wedge (y_{2m} > y_{1i})}} (\alpha + \beta - 2(x_{1i} - x_{2m})) \right] \geq 0,$$

where $\mathbf{1}(\cdot)$ is the indicator function, and the symbol $\wedge$ represents the logical conjunction "and".

The proof is shown in Appendix B.2 (in the Online Supporting Information). $\square$

The condition expressed in Proposition 5 means that the Modified Increased Spread postulate is satisfied if, for any individual i belonging to group 1, the number of individuals of group 2 who are richer than him is higher than the number of individuals of group 2 who are poorer than him, both after (see expression $\sum_m \mathbf{1}(y_{2m} > y_{1i}) - \mathbf{1}(y_{2m} < y_{1i})$) and before (see expression $\sum_m \alpha + \beta - 2 \cdot (x_{1i} - x_{2m})$) the transformation required by postulate P6*; in brief, polarization increases if there is little overlap between the two groups.

An easy-to-verify sufficient condition for property P6* is given by $m(\mathbf{x}_1) \leq m(\mathbf{x}_2)$, where $m(\mathbf{x}_1)$ and $m(\mathbf{x}_2)$ are the median incomes of the first and second groups. In other words, the polarization index P satisfies the Modified Increased Spread postulate if the medians of the two groups are ordered in the same way as their means (i.e. $\bar{x}_1 \leq \bar{x}_2$ and $m(\mathbf{x}_1) \leq m(\mathbf{x}_2)$).

Next, we investigate the postulate of Modified Increased Bipolarity (MIB) in the case of two overlapping groups.

Proposition 6 [MIB for two overlapping groups]
In the case of $k = 2$ groups, the postulate of Modified Increased Bipolarity ($P7^*$) is met by the polarization index P defined in equation 21 if

$$\left(F_1(h^*) - F_1(g^*) + \frac{1}{n_1} \right) \left(\frac{n_2}{2n} + 1 \right) > \frac{n_2}{n} \left(F_2(h^*) - F_2(g^*) \right),$$

where F_l is the cumulative distribution function of group l, with $l = 1,2$.

See the proof in Appendix B.3 (in the Online Supporting Information). $\square$

If inequality within one group decreases, then P should increase. Since this change affects both G^W and G^B, the overall effect on the polarization index P proposed in equation 21 depends on the degree of overlap between the group distributions. Intuitively, Proposition 6 requires that the majority of the individuals with income between x_{1g^*} and x_{1h^*} should belong to group 1 rather than group 2.

6. The General Case of $k \geq 2$ Overlapping Groups

Now let $k \geq 2$ groups be given to possibly overlap. We consider the same modified polarization postulates P6* (MIS) and P7* (MIB) as before. However, now the conditions under which P satisfies them are more intricate.

Proposition 7 [MIS for many overlapping groups]
Let $k \geq 2$. Then the postulate of Modified Increased Spread (P6) holds if*

$$A + B + C \geq 0,$$

with

$$A = \sum_{j \neq \{1,k\}} \sum_{i=1}^{n_1} \left(\beta \sum_{m=1}^{n_j} \left(\mathbf{1}(x_{jm} > y_{1i}) - \mathbf{1}(x_{jm} < y_{1i}) \right) + \sum_{m:(y_{1i} < x_{jm} < x_{1i})} \beta - 2 \left(x_{1i} - x_{jm} \right) \right),$$

$$B = \sum_{j \neq \{1,k\}} \sum_{i=1}^{n_k} \left(\alpha \sum_{m=1}^{n_j} \left(\mathbf{1}(x_{jm} < y_{ki}) - \mathbf{1}(x_{jm} > y_{ki}) \right) + \sum_{m:(x_{ki} < x_{jm} < y_{ki})} \alpha - 2 \left(x_{jm} - x_{ki} \right) \right),$$

$$C = \sum_{i=1}^{n_1} \left((\alpha + \beta) \sum_{m=1}^{n_k} \left(\mathbf{1}(y_{km} > y_{1i}) - \mathbf{1}(y_{km} < y_{1i}) \right) + \sum_{\substack{m:(x_{km} < x_{1i}) \\ \wedge (y_{km} > y_{1i})}} \alpha + \beta - 2 \left(x_{1i} - x_{km} \right) \right).$$

The proof is given in Appendix B.4 (in the Online Supporting Information). □

As for $k = 2$, the condition given in Proposition 7 means that the overlap must not be "too large." In particular, expressions A and B in Proposition 7 monitor the degree of overlap, respectively, between group 1 and the other groups and between group k and the other groups. Expression A is positive if the number of individuals whose income order is in accordance with the order of their group means (i.e. $x_{jm} > y_{1i}$ and $\bar{x}_j > \bar{x}_1$) is higher than the number of individuals whose income order is reversed if compared to their means ($x_{jm} < y_{1i}$ and $\bar{x}_j > \bar{x}_1$). An analogous interpretation holds for expression B. Expression C refers, instead, to the overlap between the two groups involved in the transformation, which are group 1 and group k, and it coincides with the condition required in Proposition 5 for the case of two groups.

Regarding the postulate of Modified Increasing Bipolarity, we obtain the following result:

Proposition 8 [MIB for many overlapping groups]

Let $k \geq 2$. Then the postulate of Modified Increased Bipolarity (P7) is satisfied if*

$$\left(F_1(h^*) - F_1(g^*) + \frac{1}{n_1} \right) \left(1 + \frac{n - n_1}{2n} \right) > \sum_{l=2}^{k} \frac{n_l}{n} \left(F_l(h^*) - F_l(g^*) \right),$$

where F_l is the cumulative distribution function of group l, with $l = 1, \dots, k$. The proof is given in the Appendix B.5 (in the Online Supporting Information).

The MIB postulate states that any progressive transfer occurring within any subgroup distribution should increase polarization. In particular, if the inequality within at least one group (here, w.l.o.g., we consider group 1) is reduced, this change affects both G^W and G^B. Proposition 8 says that the overall effect on the polarization index P defined in equation 21 depends on the degree of overlap between the group distributions. In particular, the polarization measure P will increase if the overlap among groups is not too high; that is, if most of the individuals whose income lies between x_{1g*} and x_{ih*} belong to the same group 1 rather than to other groups.

7. An Empirical Application

The empirical application is aimed at comparing the new polarization index with other polarization measures proposed in the literature and is focused on measuring polarization of incomes in Germany and Italy. We consider two different scenarios. First, we measure (income) bipolarization in the sense of Wolfson, meaning that two groups are formed by separating the population at the median income. In the second part of the application, we consider a situation where groups are identified not by income differences but, rather, by differences in the place of residence. This is a special case of what Gradín (2000) calls "socio-economic" polarization. We will refer to this second scenario as "regional polarization" throughout this section.

7.1. *Data*

For our application, we use data from two different sources. The first source is the EU Statistics on Income and Living Conditions (henceforth EU-SILC), provided by Eurostat. Since this survey is designed for collecting timely and comparable cross-sectional and longitudinal multidimensional microdata on income, poverty, social exclusion, and living conditions, it is perfectly suited for measuring and comparing polarization for Germany and Italy (see, e.g., Eurostat, 2007). Unfortunately, for Germany we do not have regional codes, so we are not able to measure regional polarization for Germany based on this survey. Therefore we incorporate a second source, the German Socio-Economic Panel (henceforth SOEP; see, e.g., Wagner *et al.*, 2007).[10]

For both sources, we have income data referring to all years from 2005 to 2011. We use household data and equivalized household disposable income as the variable of interest.[11] For computing regional polarization, we use the place of residence, splitting Germany into West and East and Italy into North, Central, and South; hence, we handle situations with two as well as three groups.

[10]Both sources lead to comparable results when we compute the bipolarization measures for Germany.

[11]For computing the equivalized income, we use the modified OECD scale, which assigns a value of 1 to the household head, a value of 0.5 to each additional adult, and a value of 0.3 to each child below 14 years old.

TABLE 1
INCOME POLARIZATION IN ITALY

Year	Wolfson	G^W	G^B	New	G^W_{Oka}	G^B_{Oka}	G_{Low}	G_{High}
2005	0.2673 [0.2613; 0.2734]	0.1040 [0.1017; 0.1064]	0.2193 [0.2162; 0.2224]	0.6048 [0.6014; 0.6079]	0.2080 [0.2037; 0.2125]	0.1153 [0.1132; 0.1174]	0.1966 [0.1924; 0.2010]	0.2125 [0.2066; 0.2184]
2006	0.2680 [0.2612; 0.2743]	0.1051 [0.1028; 0.1075]	0.2204 [0.2172; 0.2237]	0.6034 [0.5998; 0.6066]	0.2102 [0.2057; 0.2148]	0.1153 [0.1131; 0.1174]	0.1961 [0.1918; 0.2006]	0.2157 [0.2098; 0.2220]
2007	0.2556 [0.2501; 0.2612]	0.1017 [0.0994; 0.1039]	0.2136 [0.2105; 0.2166]	0.6057 [0.6025; 0.6090]	0.2033 [0.1992; 0.2078]	0.1119 [0.1099; 0.1139]	0.1932 [0.1890; 0.1974]	0.2074 [0.2015; 0.2135]
2008	0.2584 [0.2528; 0.2636]	0.1057 [0.1034; 0.1079]	0.2170 [0.2143; 0.2198]	0.6000 [0.5964; 0.6036]	0.2114 [0.2070; 0.2163]	0.1113 [0.1094; 0.1134]	0.1947 [0.1907; 0.1987]	0.2180 [0.2118; 0.2247]
2009	0.2552 [0.2497; 0.2610]	0.1030 [0.1007; 0.1056]	0.2141 [0.2111; 0.2172]	0.6032 [0.5994; 0.6068]	0.2061 [0.2011; 0.2115]	0.1110 [0.1089; 0.1131]	0.1944 [0.1900; 0.1987]	0.2108 [0.2042; 0.2179]
2010	0.2477 [0.2405; 0.2544]	0.1094 [0.1064; 0.1128]	0.2177 [0.2146; 0.2211]	0.5930 [0.5876; 0.5979]	0.2188 [0.2127; 0.2253]	0.1083 [0.1055; 0.1110]	0.2085 [0.2036; 0.2135]	0.2229 [0.2147; 0.2317]
2011	0.2470 [0.2402; 0.2531]	0.1105 [0.1078; 0.1133]	0.2181 [0.2151; 0.2213]	0.5911 [0.5861; 0.5957]	0.2210 [0.2157; 0.2270]	0.1076 [0.1050; 0.1103]	0.2075 [0.2030; 0.2122]	0.2263 [0.2190; 0.2345]

Note: "Wolfson" refers to the Wolfson index. G^W and G^B refer to the Gini index decomposition as in expression (5). "New" corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refer to Okamoto's decomposition as in expression (12). G_{Low} and G_{High} correspond to the Gini index of the group below the median and above the median, respectively. The values within the brackets refer to the 95 percent bootstrap confidence intervals.

TABLE 2
INCOME POLARIZATION IN GERMANY

Year	Wolfson	G^W	G^B	New	G^W_{Oka}	G^B_{Oka}	G_{Low}	G_{High}
2005	0.2089 [0.2013; 0.2190]	0.0985 [0.0953; 0.1014]	0.1902 [0.1863; 0.1947]	0.5961 [0.5916; 0.6002]	0.1972 [0.1913; 0.2031]	0.0915 [0.0884; 0.0947]	0.1525 [0.1486; 0.1568]	0.2173 [0.2093; 0.2257]
2006	0.2095 [0.2034; 0.2211]	0.0977 [0.0934; 0.1012]	0.1896 [0.1855; 0.1949]	0.5976 [0.5918; 0.6025]	0.1954 [0.1883; 0.2026]	0.0919 [0.0892; 0.095]	0.1491 [0.1453; 0.1536]	0.2163 [0.2067; 0.2265]
2007	0.2107 [0.2019; 0.2236]	0.0976 [0.0926; 0.1016]	0.1894 [0.151; 0.1949]	0.5971 [0.5898; 0.6029]	0.1956 [0.1878; 0.2043]	0.0915 [0.0885; 0.0951]	0.1507 [0.1465; 0.1561]	0.2159 [0.2052; 0.2281]
2008	0.2173 [0.2059; 0.2283]	0.0938 [0.0907; 0.0966]	0.1878 [0.1834; 0.1930]	0.6032 [0.5987; 0.6077]	0.1881 [0.1823; 0.1941]	0.0935 [0.0902; 0.0970]	0.1486 [0.1450; 0.1526]	0.2063 [0.1981; 0.2146]
2009	0.2282 [0.2138; 0.2374]	0.0924 [0.0894; 0.097]	0.1929 [0.1872; 0.1977]	0.6043 [0.5996; 0.6089]	0.1878 [0.1813; 0.1949]	0.0975 [0.0934; 0.1008]	0.1523 [0.1475; 0.1565]	0.2049 [0.1956; 0.2145]
2010	0.2318 [0.2215; 0.2401]	0.0929 [0.090; 0.0965]	0.1941 [0.1897; 0.1984]	0.6056 [0.6009; 0.6098]	0.1881 [0.1827; 0.1939]	0.0989 [0.0959; 0.1017]	0.1541 [0.1503; 0.1581]	0.2040 [0.1962; 0.2121]
2011	0.2192 [0.2093; 0.2271]	0.0939 [0.0904; 0.0996]	0.1905 [0.1864; 0.1941]	0.6042 [0.5953; 0.6111]	0.1886 [0.1812; 0.1989]	0.0958 [0.0917; 0.0990]	0.1512 [0.1471; 0.1549]	0.2057 [0.1953; 0.2203]

Note: "Wolfson" refers to the Wolfson index. G^W and G^B refer to the Gini index decomposition as in expression (5). "New" corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refer to Okamoto's decomposition as in expression (12). G_{Low} and G_{High} correspond to the Gini index of the group below the median and above the median, respectively. The values within the brackets refer to the 95 percent bootstrap confidence intervals.

7.2. *Bipolarization*

We first consider the more familiar scenario of (income) bipolarization as in Wolfson's approach. We apply both the Wolfson and the new polarization measure to German and Italian data, splitting each population into two groups by the country-specific income median. This allows us to compare the two countries and the two measures as well. To get a closer insight into the two measures, we also report the corresponding Gini index decompositions, as well as the Gini index of the low- and the high-income groups (see Tables 1 and 2). We also report 95 percent bootstrap confidence intervals, which have been obtained from year 2000 bootstrap replicates of the original samples. Since we are in the special situation of two equally sized groups, we can observe the relation between the two within-group components, namely that the within-group component of the Wolfson measure is half the within-group component of the new measure.

With respect to measuring (income) bipolarization, the two polarization measures (the Wolfson's and the new index proposed) show similar results. In particular, according to both measures, income bipolarization in Italy has significantly decreased from 2006 to 2012 (see Table 1). On the contrary, in Germany polarization has not changed significantly over the period under consideration; this result does not depend on the polarization measure employed.

Looking at the Wolfson measure, levels of polarization are much higher in Italy than in Germany for all years. The new polarization measure, instead, registers much smaller differences between the two countries, although the direction of change is the same. Looking, moreover, at the within- and between-group Gini index, we note that the overall inequality is also higher in Italy than in Germany.

7.3. *Regional Polarization*

We now consider a situation in which identification and alienation are not based solely on income disparities but also on differences in the place of residence. In this context, the Wolfson and any other bipolarization measure are of no use, so we will compare the new measure with the index P^{LU} proposed by Lasso de la Vega and Urrutia (2006), defined as in expression (3). For the latter, two different settings for the parameters α and β will be considered.[12]

Before performing this second analysis, one should verify that the conditions required by the Modified Increased Spread and the Modified Increased Polarity properties (see Propositions 5–8) are satisfied in this context. The Modified Increased Spread property holds both for Germany and for Italy in all years, while the Modified Increased Polarity property may depend on the income levels and on the groups of the two individuals involved in the transfer. We have simulated all possible transfers and in all cases the condition holds for Italy as well as for

---

[12]For K, we have chosen $K = k^{2 + \alpha}/[2(k-1) \log (k \cdot \mu - k + 1)]$, with k being the number of groups. Assuming that the minimal mean income of a group is one—keep in mind that the measure is not defined for mean incomes of zero—this choice of K normalizes the measure between 0 and 1. Therefore K depends on the number of groups and differs between Germany and Italy.

TABLE 3
REGIONAL POLARIZATION IN ITALY

Year	P^{LU}		New	G^W_{Oka}	G^B_{Oka}	G_{North}	$G_{Central}$	G_{South}
	$\alpha = \beta = 1$	$\alpha = 1.6, \beta = 1$						
2005	0.0244 [0.0224; 0.0264]	0.0267 [0.0245; 0.0289]	0.3841 [0.3788; 0.3892]	0.3129 [0.3074; 0.3183]	0.0113 [0.010; 0.0128]	0.3019 [0.2942; 0.3092]	0.3140 [0.3022; 0.3266]	0.3357 [0.3262; 0.3450]
2006	0.0235 [0.0214; 0.0256]	0.0257 [0.0233; 0.0280]	0.3839 [0.3224; 0.3893]	0.3140 [0.3086; 0.3198]	0.0115 [0.0101; 0.013]	0.3024 [0.2952; 0.3096]	0.3218 [0.3083; 0.3370]	0.3321 [0.3217; 0.3440]
2007	0.0225 [0.0207; 0.0244]	0.0245 [0.0226; 0.0267]	0.3891 [0.3269; 0.3939]	0.3042 [0.2987; 0.3095]	0.0104 [0.0092; 0.0117]	0.2948 [0.2875; 0.3021]	0.3090 [0.2961; 0.3233]	0.3200 [0.3113; 0.3282]
2008	0.0238 [0.0220; 0.0256]	0.0260 [0.0240; 0.0280]	0.3871 [0.3821; 0.3917]	0.3089 [0.3040; 0.3136]	0.0112 [0.010; 0.0127]	0.2987 [0.2913; 0.3058]	0.3158 [0.306; 0.3260]	0.3251 [0.3171; 0.3333]
2009	0.0228 [0.0210; 0.0246]	0.0250 [0.0229; 0.0270]	0.3880 [0.3825; 0.3934]	0.3058 [0.3001; 0.3112]	0.0097 [0.0084; 0.011]	0.2991 [0.2909; 0.3076]	0.3052 [0.2945; 0.3162]	0.3202 [0.3122; 0.3282]
2010	0.0256 [0.0236; 0.0276]	0.0281 [0.0258; 0.0304]	0.3872 [0.3816; 0.3926]	0.3102 [0.3044; 0.3154]	0.0124 [0.011; 0.0139]	0.2978 [0.2898; 0.3056]	0.3147 [0.3042; 0.3258]	0.3338 [0.3226; 0.3453]
2011	0.0247 [0.0229; 0.0266]	0.0270 [0.0250; 0.0291]	0.3857 [0.3808; 0.3907]	0.3118 [0.3067; 0.3169]	0.0115 [0.0102; 0.013]	0.3004 [0.2937; 0.3077]	0.3182 [0.3082; 0.3284]	0.3312 [0.3220; 0.3400]

Note: P^{LU} refers to the index of Lasso de la Vega and Urrutia (2006). "New" corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refer to Okamoto's decomposition as in expression (12). G_{North}, $G_{Central}$, and G_{South} correspond to the Gini index of North, Central, and South Italy, respectively. The values within the brackets refer to the 95 percent bootstrap confidence intervals.

TABLE 4
REGIONAL POLARIZATION IN GERMANY

Year	P^{LU} $\alpha = \beta = 1$	P^{LU} $\alpha = 1.6, \beta = 1$	New	G^W_{Oka}	G^B_{Oka}	G_{West}	G_{East}
2005	0.0143 [0.0125; 0.0161]	0.0165 [0.0145; 0.0185]	0.2962 [0.2835; 0.3088]	0.2783 [0.2715; 0.2854]	0.0041 [0.0032; 0.0051]	0.2861 [0.2783; 0.2942]	0.2403 [0.2300; 0.2517]
2006	0.0148 [0.0129; 0.0167]	0.0171 [0.0150; 0.0193]	0.2965 [0.2831; 0.3104]	0.2769 [0.2691; 0.2861]	0.0045 [0.0035; 0.0056]	0.2850 [0.2760; 0.2957]	0.2371 [0.2267; 0.2482]
2007	0.0149 [0.0131; 0.0168]	0.0173 [0.0152; 0.0194]	0.2931 [0.2791; 0.3067]	0.2763 [0.2680; 0.2856]	0.0045 [0.0036; 0.0057]	0.2837 [0.2739; 0.2945]	0.2390 [0.2293; 0.2488]
2008	0.0140 [0.0121; 0.0159]	0.0163 [0.0141; 0.0185]	0.2944 [0.2805; 0.3095]	0.2718 [0.2642; 0.2798]	0.0043 [0.0033; 0.0055]	0.2783 [0.2693; 0.2875]	0.2400 [0.2281; 0.2528]
2009	0.0136 [0.0115; 0.0157]	0.0159 [0.0135; 0.0184]	0.2819 [0.2677; 0.2974]	0.2775 [0.2689; 0.2870]	0.0040 [0.0031; 0.0051]	0.2833 [0.2734; 0.2943]	0.2477 [0.2342; 0.2626]
2010	0.0139 [0.0122; 0.0155]	0.0161 [0.0142; 0.0180]	0.2933 [0.2809; 0.3064]	0.2797 [0.2730; 0.2869]	0.0039 [0.0031; 0.0049]	0.2876 [0.2797; 0.2959]	0.2415 [0.2323; 0.2508]
2011	0.0129 [0.0111; 0.0146]	0.0149 [0.0129; 0.0168]	0.2761 [0.2865; 0.3127]	0.0036 [0.2695; 0.2826]	0.2815 [0.0028; 0.0046]	0.2509 [0.2740; 0.2889]	[0.2398; 0.2614]

Note: P^{LU} refers to the index of Lasso de la Vega and Urrutia (2006). "New" corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refer to Okamoto's decomposition as in expression (12). G_{West} and G_{East} correspond to the Gini index of West and East Germany, respectively. The values within the brackets refer to the 95 percent bootstrap confidence intervals.

Germany. (The data are not shown here, but are available from the authors upon request.)

Estimates of both polarization measures (P^{LU} and the new index proposed) are shown in Tables 3 and 4, again accompanied by associated bootstrap confidence intervals and by the within- and between-group components of Okamoto's Gini decomposition. We first notice that for both countries the values of the new measure are now lower than in the previous analysis of income bipolarization. This is due to the fact that the inequality within regional groups is much higher than in the groups divided by the median income, while the income disparities among regional groups are much lower than the disparities across the income groups. This seems plausible since in this second analysis we allow the groups to overlap, so the difference between groups can potentially be much lower.[13]

Tables 3 and 4 clearly show that regional polarization is significantly higher in Italy than in Germany for all years, according to both Lasso de la Vega and Urrutia (2006) and to the new measure. This is mostly due to the fact that the inequality within the Italian macro-regions (North, Central, and South) is higher. The between-group component is quite low in both countries, meaning that there is a high degree of overlap across the regional income distributions. Over time, both polarization measures reveal no significant changes in regional polarization in Italy and in Germany.

8. Concluding Remarks

In this paper, we have proposed a new polarization measure that resembles the structure of the polarization index proposed by Wolfson (1994), while simultaneously reflecting the ideas of identification and alienation introduced by Esteban and Ray (1994). The new polarization index is based on a recent decomposition of the Gini index by Okamoto (2009), which divides the overall inequality into a within-group and a between-group component. By making use of this decomposition, the new index maintains the simplicity and immediate comprehension of the Wolfson index, while being much more flexible.

The increased flexibility refers to two aspects of polarization measurement: (i) the new index allows for the presence of more than two groups, and (ii) the groups may also overlap. The new approach, therefore, accommodates both researchers who are interested in measuring the traditional income bipolarization and those who assume that the population is made up of more than two groups, even if the groups are formed by characteristics other than income—a situation that generally introduces some overlap between the groups. The empirical application has shown how the new approach enriches the analysis based on standard measures of income polarization and how it contributes to disentangling the different faces of polarization analysis.

---

[13]If the overlap is very large, the between-group component can even tend to zero, independent of the overall or within-group inequality.

Finally, similar to socioeconomic inequality, polarization may be seen as a multivariate phenomenon that takes account of not only income but also attributes such as wealth, education, and health variables. Okamoto (2009) has provided analogous decompositions for two multidimensional extensions of the Gini index, such as the Distance Gini index and the Volume Gini index (Koshevoy and Mosler, 1997). Multivariate polarization indices can be based on these decompositions and developed along the lines of Gigliarano and Mosler (2009). However, this is a matter for future research.

References

Anderson, G., "Towards an Empirical Analysis of Polarization," *Journal of Econometrics*, 122, 1–26, 2004.

Anderson, G., Y. Ge, and L. Wah, "Distributional Overlap: Simple, Multivariate, Parametric and Non-parametric Tests for Alienation, Convergence and General Distributional Difference Issues," *Econometric Reviews*, 29, 247–75, 2009.

Anderson, T. W., "On the Distribution of the Two-Sample Cramer – von Mises Criterion," *Annals of Mathematical Statistics*, 33, 1148–59, 1962.

Bhattacharya, N. and B. Mahalanobis, "Regional Disparities in Household Consumption in India," *Journal of the American Statistical Association*, 62, 143–61, 1967.

Bossert, W. and W. Schworm, "A Class of Two-Group Polarization Measures," *Journal of Public Economic Theory*, 10, 1169–87, 2008.

Chakravarty, S., *Inequality, Polarization and Poverty: Advances in Distributional Analysis*, Springer-Verlag, Berlin, 2009.

Chakravarty, S. and C. D'Ambrosio, "Polarization Orderings of Income Distributions", *Review of Income and Wealth*, 56, 47–64, 2010.

Chakravarty, S., A. Majumder, and S. Roy, "A Treatment of Absolute Indices of Polarization," *Japanese Economic Review*, 58, 273–93, 2007.

Chakravarty, S. R. and A. Majumder, "Inequality, Polarization and Welfare: Theory and Applications," *Australian Economic Papers*, 40, 1–13, 2001.

Dagum, C., "A New Approach to the Decomposition of the Gini Income Inequality Ratio," *Empirical Economics*, 22, 515–31, 1997.

D'Ambrosio, C., "Household Characteristics and the Distribution of Income in Italy: An Application of Social Distance Measures," *Review of Income and Wealth*, 47, 43–64, 2001.

Deutsch, J., M. Hanoka, and J. Silber, "On the Link Between the Concepts of Kurtosis and Bipolarization," *Economics Bulletin*, 4, 1–6, 2007.

Donaldson, D. and J. Weymark, "A Single-Parameter Generalization of the Gini Indices of Inequality," *Journal of Economic Theory*, 22, 67–86, 1980.

Duclos, J., J. Esteban, and D. Ray, "Polarization: Concepts, Measurement, Estimation," *Econometrica*, 72, 1737–72, 2004.

Ebert, U., "The Decomposition of Inequality Reconsidered: Weakly Decomposable Measures," *Mathematical Social Sciences*, 60, 94–103, 2010.

Esteban, J. and D. Ray, "On the Measurement of Polarization," *Econometrica*, 62, 819–51, 1994.

Esteban, J., C. Gradín, and D. Ray, "An Extension of a Measure of Polarization, with an Application to the Income Distribution of Five OECD Countries," *Journal of Economic Inequality*, 5, 1–19, 2007.

Eurostat, "Comparative EU Statistics on Income and Living Conditions: Issues and Challenges," *Proceedings of the EU-SILC conference (Helsinki, 6–8 November 2006), Methodologies and Working Papers*, Eurostat, Luxembourg, 2007.

Gigliarano, C. and K. Mosler, "Constructing Indices of Multivariate Polarization," *Journal of Economic Inequality*, 7, 435–460, 2009.

Gradín, C., "Polarization by Sub-populations in Spain, 1973–91," *Review of Income and Wealth*, 46, 457–74, 2000.

Koshevoy, G. and K. Mosler, "Multivariate Gini indices," *Journal of Multivariate Analysis*, 60, 252–76, 1997.

Lasso de la Vega, M. and A. Urrutia, "An Alternative Formulation of the Esteban–Gradin–Ray Extended Measure of Bipolarization," *Journal of Income Distribution*, 15, 42–54, 2006.

Lasso de la Vega, M., H. Diez, and A. Urrutia, "Unit Consistency and Bipolarization of Income Distributions," *Review of Income and Wealth*, 56, 65–83, 2010.

Okamoto, M., "Decomposition of the Gini and Multivariate Gini Indices," *Journal of Economic Inequality*, 7, 153–77, 2009.

Pyatt, G., "On the Interpretation and Disaggregation of Gini Coefficients," *Economic Journal*, 86, 243–55, 1976.

Rodríguez, J. and R. Salas, "Extended Bi-polarization and Inequality Measures," *Research on Economic Inequality*, 9, 69–83, 2003.

Shorrocks, A., "The Class of Additively Decomposable Inequality Measures," *Econometrica*, 48, 613–25, 1980.

Silber, J., J. Deutsch, and M. Hanoka, "On the Link Between the Concepts of Kurtosis and Bipolarization," *Economics Bulletin*, 4, 1–6, 2007.

Wagner, G. G., J. R. Frick, and J. Schupp, "The German Socio-Economic Panel study (SOEP)— Evolution, Scope and Enhancements," *Journal of Applied Social Science (Schmollers Jahrbuch)*, 127, 139–69, 2007.

Wang, Y. and K. Tsui, "Polarization Orderings and New Classes of Polarization Indices," *Journal of Public Economic Theory*, 2, 349–63, 2000.

Wolfson, M. C., "Divergent Inequalities: Theory and Empirical Results," *Review of Income and Wealth*, 43, 401–21, 1997.

———, "When Inequalities Diverge," *American Economic Review*, 48, 353–8, 1994.

Yitzhaki, S. and R. I. Lerman, "Income Stratification and Income Inequality," *Review of Income and Wealth*, 37, 313–29, 1991.

Zhang, X. and R. Kanbur, "What Difference Do Polarisation Measures Make? An Application to China," *Journal of Development Studies*, 37, 85–98, 2001.

Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendices

A. Proof of Minimum and Maximum Value of D = GB-GW for Two Generic Groups

B. Proofs of the Properties

B.1. Increased Spread for Two Non-overlapping Groups

B.2. Modified Increased Spread for Two Overlapping Groups

B.3. (Modified) Increased Bipolarity for Two Groups (Overlapping and Not)

B.4. Modified Increased Spread for k>=2 Generic Groups

B.5. Modified Increased Polarity for k>=2 Generic Groups