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MULTILATERAL INDICES: CONFLICTING APPROACHES?

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This short paper focusses on an apparent conflict between two results from different approaches to the problem of finding multilateral index numbers. The impossibility theorem of Van Veelen (2002) is an axiomatic result that rules out the existence of a multilateral index that satisfies four modest requirements. This also implies that no bilateral index can consistently be generalized to a multilateral setting. Adopting a revealed preference approach, Dowrick and Quiggin (1997) however construct a multilateral extention of Fisher's ideal index, which preserves a range of desirable properties. This note shows what it is that drives the divergence between those two results. It also gives implications for practical use of results from either approach.

1. INTRODUCTION

There are a few different ways of trying to find multilateral index numbers that permit comparisons of welfare levels between a number of countries or a number of time periods. One option is to start with formulating sensible properties that indices should have and then determine which indices, if any, do have these properties. This is the axiomatic approach. Another possibility is to assume that prices and quantities are observations from a common preference relation held by a representative individual and then find indices that respect revealed preferences.

In this short paper, we will look at two relatively recent contributions, one from either approach, that seem to produce contradictory results. Van Veelen (2002) formulates four modest requirements and proves that there is no multilateral index that satisfies all four of them. This implies that Fisher's (1922) ideal index, that does satisfy these requirements for comparison between two countries only, cannot be consistently generalized to a multilateral setting. By contrast, Dowrick and Quiggin (1997) derive, following Afriat (1967, 1981), Diewert (1973) and Varian (1983), a multilateral extension of the Fisher ideal index, which preserves a range of desirable properties. Thus, there appears to be a conflict between the axiomatic approach, adopted by Van Veelen, and the revealed preference approach, adopted by Dowrick and Quiggin. The purpose of this note is to resolve

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this apparent contradiction. We will pinpoint what exactly creates the paradox and we will elaborate on the implications for practical use.

2. Contradiction

Van Veelen's (2002) Apples and Oranges theorem amounts to the following. Prices and quantities of M goods that are being consumed in K countries are known and ordered in $K \times M$ matrices P and Q. The k-th rows of matrices P and Q, corresponding to the prices and quantities for country k, are denoted by vectors p^k, $q^k \in \mathbb{R}_+^M$. A mapping $F : \mathbb{R}_+^{2\times K\times M} \to \mathbb{R}_+^K$ represents a way of comparing the consumptionbundles of these K countries, where $F_k(P, Q) > F_l(P, Q)$ implies that F ranks country k above country l on the basis of the data set (P, Q). Van Veelen imposes four axioms (*Weak Continuity*, *Dependence on Prices*, the *Weak Ranking Restriction* and *Independence of Irrelevant Countries*) and proves that for $K > 2$ no such mapping F is consistent with the stated axioms. Please note that formulating the index number problem as a search for an appropriate function F that maps prices and quantities to a vector with K entries already implies that index numbers are transitive, while alternative ways could in principle allow for intransitivity (see for example, Neary, 2004).

The revealed preference approach of Dowrick and Quiggin (1997), following results from Afriat (1967, 1981), Diewert (1973) and Varian (1983), starts with the assumption that the data are observations from a preference relation that can be represented by a non-satiated utility function. If the data do not contradict this assumption, then indices are computed that reflect a utility function that rationalizes the data. This implies that data sets are restricted to those that satisfy the Generalized Axiom of Revealed Preferences (GARP) or, even better, the Homothetic Axiom of Revealed Preference (HARP), which means that there is a *homothetic* utility function that rationalizes the data. If there is indeed a homothetic utility function that rationalizes the data, then the magnitude of these indices can be naturally informative, but even if the utility function is not homothetic, then it gives at least a ranking that is in line with a possible preference relation that rationalizes the data. In short, it is a way of recovering, or reconstructing, one single preference relation that presumably lies behind the data.

While Van Veelen's negative result implies that Fisher's (1922) ideal index cannot be consistently generalized to a multilateral setting, Dowrick and Quiggin have nonetheless constructed indices that are perfectly acceptable in a revealed preference context. In order to understand how this apparent contradiction can arise, we will first copy some definitions and notation from Varian (1983, 1992).

Notation 1: We write $q^i R^D q$ if observation q^i is directly revealed preferred to a consumption bundle q, that is, if $p^i q^i \geq p^i q$.

Notation 2: We write $q^i P^D q$ if observation q^i is strictly directly revealed preferred to a consumption bundle q, that is, if $p^i q^i > p^i q$.

Notation 3: We write $q^i R q$ if observation q^i is revealed preferred to a consumption bundle q, that is, if there is some sequence of bundles $q^j, q^k, \ldots, q^l$ such that $q^i R^D q^j$, $q^j R^D q^k$, $\ldots$, $q^l R^D q$.

Definition 4: The data set (P, Q) satisfies the Generalized Axiom of Revealed Preference if q^iRq^j implies not $q^jP^Dq^i$.

The set of datasets (P, Q) that satisfy GARP will be denoted by $D_G \subset \mathbb{R}_+^{2\times K\times M}$ and we will write $D_H \subset D_G$ for the set of datasets (P, Q) that satisfy HARP (see Varian (1983) for the definition). The approach of Dowrick and Quiggin restricts datasets to those that satisfy GARP or HARP, which produces indices that imply a function F that is defined on D_G an D_H respectively, rather than on the whole set $\mathbb{R}_+^{2\times K\times M}$. It should be noted that restricting the domain to either one of these sets does not reverse the impossibility result of Van Veelen; inspection of the proof reveals that the argument applies equally when all data sets are elements of D_H.

We will also need an axiom that states when a function F respects revealed preference. This could be formulated in a few different ways, but we will see that this one will do.

Axiom 5 (Consistency with Revealed Preference)
$F : D_G \to \mathbb{R}_+^K$ is consistent with revealed preference if
$q^iR^Dq^j \Rightarrow F_i(P, Q) \geq F_j(P, Q) \quad \forall (P, Q) \in D_G$

Furthermore we reproduce two axioms from Van Veelen (2002)

Axiom 6 (Independence of Irrelevant Countries)
If for matrices P, Q, R and S, $r^k = p^k$, $r^l = p^l$, $s^k = q^k$ and $s^l = q^l$, then $F_k(P, Q) > F_l(P, Q) \Leftrightarrow F_k(R, S) > F_l(R, S)$.

Axiom 7 (Weak Ranking Restriction)
$q^k > q^l \Rightarrow F_k(P, Q) > F_l(P, Q)$,
where $q^k > q^l$ means that $q_j^k > q_j^l$, $j = 1, \ldots, M$.

The core of the conflict will show up if we focus on datapoints with characteristics that feature in the following Lemma. Figures 1 and 2 illustrate the construction of p', q' and p'', q''.

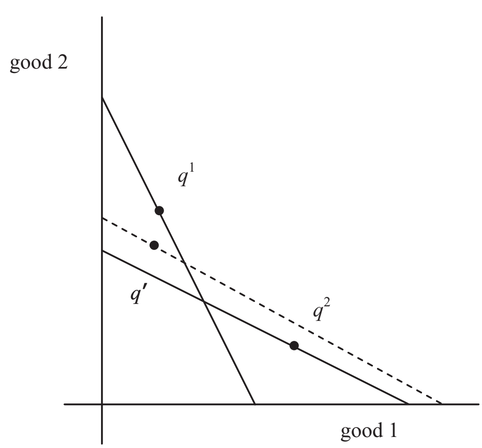

Figure 1

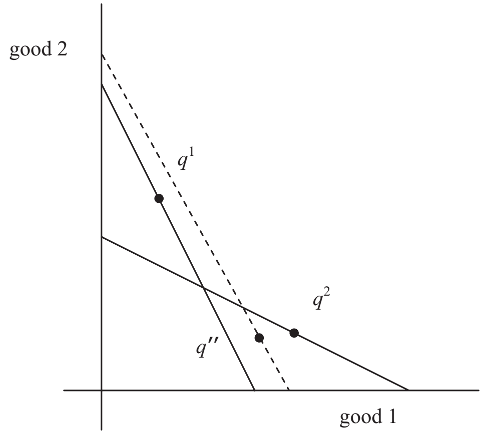

Figure 2

Lemma 8: Take $M = 2$ and let p^1, q^1, p^2 and $q^2 \in \mathbb{R}_+^M$ be such that $p^1q^2 > p^1q^1$ and $p^2q^1 > p^2q^2$. Then there exist p', q', p'' and q'' such that

$$\begin{matrix} q^1 P^D q' \text{ and not } q' R^D q^1 \\ q' P^D q^2 \text{ and not } q^2 R^D q' \end{matrix} \quad \text{and} \quad \begin{matrix} q^2 P^D q'' \text{ and not } q'' R^D q^2 \\ q'' P^D q^1 \text{ and not } q^1 R^D q'' \end{matrix}$$

Proof: Suppose without loss of generality that $q_1^1 < q_1^2$. Then define for $\delta > 0$

$$\begin{matrix} p' = \left[(1-\delta)\, p_1^2,\, p_2^2\right] & q' = \left[(1-2\delta)\, q_1^1,\, (1-\delta)\, q_2^1\right] \\ p'' = \left[p_1^1,\, (1-\delta)\, p_2^1\right] & q'' = \left[(1-\delta)\, q_1^2,\, (1-2\delta)\, q_2^2\right] \end{matrix}$$

and take δ suitably small. This gives the required p', q' and p'', q''. ■

Even simpler constructions of p', q' and p'', q'' are also possible, but this one has an extra feature, which is that the datasets we will make using them will not only satisfy GARP, but HARP as well.

If we create datasets by starting with p^1, q^1 and p^2, q^2 and adding p', q' resp. p'', q'' as a third datapoint, then we have enough to prove the following proposition:

Proposition 9: There is no function $F : D_G \to \mathbb{R}_+^K$ that satisfies axioms 5, 6 and 7 with $M \geq 2$ and $K \geq 3$.

Proof: Assume that there is a function $F : D_G \to \mathbb{R}_+^K$ that does satisfy axioms 5, 6 and 7 with $M = 2$ and $K = 3$. Then take P, Q, R and S as follows:

$$P = \begin{bmatrix} p^1 \\ p^2 \\ p' \end{bmatrix}, \quad Q = \begin{bmatrix} q^1 \\ q^2 \\ q' \end{bmatrix}, \quad R = \begin{bmatrix} p^1 \\ p^2 \\ p'' \end{bmatrix} \text{ and } S = \begin{bmatrix} q^1 \\ q^2 \\ q'' \end{bmatrix}$$

Both these data sets satisfy GARP and therefore fall within the domain of F. Now the weak ranking restriction (7) requires that $F_1(P, Q) > F_3(P, Q)$, while consistency with revealed preference (5) requires $F_3(P, Q) \geq F_2(P, Q)$, which amounts to $F_1(P, Q) > F_2(P, Q)$. On the other hand, the weak ranking restriction demands $F_2(R, S) > F_3(R, S)$ and consistency with revealed preference requires $F_3(R, S) \geq F_1(R, S)$, which implies that $F_2(R, S) > F_1(R, S)$. This however contradicts independence of irrelevant countries (6).

Extension to cases with larger M and K is obvious. ■

Note that if we leave out the weak ranking restriction, then $F \equiv [1, 1, 1]^T$ would satisfy the other two axioms. Without the independence of irrelevant countries, utility levels from utility functions that rationalize the data set make an F that satisfies the remaining two demands. If we disregard consistency with revealed preference, an example of a function that does not violate the rest is one with components $F_k(P, Q) = \langle \bar{p}, q^k \rangle$ in which $\bar{p}$ is a fixed weight vector. Finally the weak ranking restriction together with consistency with revealed preference could in this proposition be replaced by a stronger version of the latter:

Axiom 10 (Consistency with Strict Revealed Preference)
$F : D_G \to \mathbb{R}_+^K$ is consistent with strict revealed preference if it is consistent with revealed preference and if $q^i P^D q^j \Rightarrow F_i(P, Q) > F_j(P, Q) \quad \forall (P, Q) \in D_G$.

3. Interpretation

What Proposition 9 shows is that consistency with revealed preference cannot be reconciled with independence of irrelevant countries (unless we give up the weak ranking restriction). Indices that follow from a revealed preference approach obviously have the first property, while the latter features in the axiomatic approach. If we want to understand how the difference between the two approaches arises, it is very instructive to look at what they do with the data sets we constructed for the proof of the proposition.

The revealed preference approach starts from the assumption that the preferences of the representative consumer are the same in all three countries we consider. Therefore it takes the data from the third country as useful information about the one (representative) utility function that, by hypothesis, these countries share. Consequently it is natural that data from a third country should affect whether country 1 ranks higher than country 2 or vice versa. If the assumption is correct and the data points from different countries are indeed observations from one single preference relation, then we can state that independence of irrelevant countries does rule out perfectly reasonable index numbers.

The axiomatic approach, on the other hand, allows for all countries to have a representative utility function of their own. In fact, it does not make any assumptions about consumer behavior at all. When comparing consumption bundles in two countries, prices and quantities from a third country are therefore not considered to be informative and it is considered undesirable if they make a difference for how the first two countries rank relative to each other.

The great attraction of the representative consumer assumption and the even stronger assumption that the representative utility function is homothetic is that it permits a sensible economic interpretation to be given to statements that are commonly made using index numbers. Consider for example the statement "Consumption per person is 10 percent higher in country A than in country B" or the equivalent statement, for the time series case, "Consumption per person grew by 10 percent between time period 1 and time period 2." If the index of consumption is derived from a common homothetic utility function, these statements can be rephrased as "The consumption bundle in which all quantities for country B (or time period 1) are increased by 10 percent yields the same utility as the consumption bundle for country A (time period 2)." With common, but not homothetic preferences, there is a natural economic interpretation for ordinal statements such as "Consumption per person is higher in country A (or time period 1) than in country B (or time period 2)."

By contrast, the axiomatic approach could be seen as an effort to find indices that still make sense if we do away with assumptions concerning preferences in different countries or in different periods. This would imply that if the assumptions turn out not to hold, we could fall back on such indices. In a setting where countries may have different representative consumers, it is natural to demand independence of irrelevant countries. The idea behind it came from Fisher (1922) and his book gives ample reasons why it makes sense to require that an index number should have this property.

Perhaps *the* important question is therefore whether or not the representative (homothetic) consumer assumption holds. If we encounter violations of GARP, then that indicates that it does not hold, which frustrates a revealed preference approach. However, if the data do pass GARP, then this only means that the assumption is not proven wrong by the observations. It is not necessarily informative about how likely it is that the assumption actually is correct. In handling the data, we should be aware of this. For instance if we find violations of GARP, we should resist the temptation to simply cross countries off the list until we do have a set of data that is rationalizable. Another look at the two data matrix pairs from Proposition 9 is also instructive. If the difference between them is caused by the fact that one of them consists of prices and quantities in, say, Germany, France and the U.K., while the other consists of prices and quantities in Germany, France and Italy, then it is obviously very dubious to stick to the revealed preference approach. After all, the data from Germany, France, the U.K. and Italy together imply that at least they are not all four of them observations from one utility function. (Of course the same goes if we substitute years for countries.)

Neary (2004) suggests a way around this issue by selecting a *reference* consumer, rather than assuming that there exists a representative one. The question that is answered in that paper is therefore: "How well off would the same reference consumer be in different countries?" From the multitude of candidate reference consumers, he chooses a hypothetical consumer whose consumption patterns mimic world consumption behavior as closely as possible.

Finally, it should also be noted that with any finite data set generated by a common utility function, there will exist a range of indeterminacy, since the same choices are consistent with a range of possible preferences. Therefore the revealed preference approach in itself only yields bounds on admissible values of real income indices. Hence, if the object is to make statements with a natural and defensible economic interpretation, it is normally appropriate to give a range of possible values rather than a point estimate (see Dowrick and Quiggin (1997) for some examples). Yet practical uses of index numbers, such as adjustments to wage contracts and pension payments, frequently require the use of a point estimate. The desirable properties of such an estimate depend on the purpose for which the estimate is to be applied, and do not necessarily depend on the existence of a rigorously defensible economic interpretation, and in particular not on common preferences.

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