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AN UPPER BOUND OF THE GINI INDEX IN THE ABSENCE OF MEAN INCOME INFORMATION

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In this paper, an upper bound of the Gini index, based on grouped data, is proposed assuming that there is no information on all the group mean incomes as well as the overall mean income but the limits of the income brackets are known. An important advantage of this proposal is that conventional formulas for the upper bound of the Gini index could be applied directly by substituting the (unknown) mean income for each income bracket with the corresponding value that maximizes the grouping correction for that income bracket. The effects of varying the number and size of income brackets are investigated.

1. Introduction

For various reasons, most official income distribution statistics are published in grouped form with a selected number of non-overlapping income brackets. This grouping results in downward biased point estimates of the Gini index if piecewise linear approximations to the associated Lorenz curves (LCs) at the observed points are used. The resulting bias may complicate inter-temporal comparisons of income inequality, leading to spurious inferences regarding inequality trends, as noted by Petersen (1979), Lerman and Yitzhaki (1989), and Deltas (2003).

The aforementioned limitations of piecewise linear approximations to LCs have led to the search for functional forms that satisfy all the properties of a theoretical LC (i.e. it is twice differentiable, convex, monotonically increasing, and passes through the points (0,0) and (1,1)). The point estimates of the Gini index, and in some cases the underlying density, are then deduced from those of the estimated parameters of such functional forms. It is well known (e.g. Ogwang and Rao, 1996, 2000; Sarabia *et al.*, 2005) that estimates of the Gini index are sensitive to the choice of the functional form of the underlying LC. Furthermore, empirical studies by Rao and Tam (1987), Villaseñor and Arnold (1989), Chotikapanich (1993), Schader and Schmid (1994), Wan (1999), Cheong (2002), and Sarabia *et al.* (2005), among others, highlight the lack of consensus regarding the most appropriate functional form for the LC.

As an alternative to obtaining point estimates of the Gini index, Gastwirth (1972), Mehran (1975), Murray (1978), Fuller (1979), Giorgi and Pallini (1987), Silber (1990), Ogwang (2003), and Ogwang and Wang (2004), among others, have derived lower and upper bounds from grouped data, within which the Gini index must lie regardless of the functional form of the underlying distribution of income.

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TABLE 1

Information Requirements for Computing the Upper Bound of the Gini Index*

Proposal	Mean Incomes: Group	Mean Incomes: Overall	Limits of Income Brackets	No. Income-Receiving Units
Gastwirth (1972)	R	R	R	N
Mehran (1975)	N	N	N	N
Murray (1978)	N	R	R	R
Fuller (1979)	N	R	R	N
Giorgi and Pallini (1987)	R	R	N	R
Silber (1990)	N	N	N	N
Ogwang (2003)	R/N	R/N	R	N
Ogwang and Wang (2004)	R/N	R/N	R	N
Present proposal	N	N	R	N

Notes: *R denotes required information; N denotes not required; and R/N denotes may be required or not required depending on the available mean income information.

The lower bound assumes that the incomes in each income bracket are equally distributed. The upper bound also incorporates a "grouping correction" (GC) that generally assumes maximum inequality in each bracket.

The aforementioned bound proposals have been widely used by applied researchers either to determine the range of values within which the Gini index for a particular country/region lies or to compare alternative LC models (e.g. Kakwani and Podder, 1976; Cowell and Mehta, 1982; Schader and Schmid, 1994; Abdalla and Hassan, 2004).

In practice, the size of the estimated bounds depends on the number of groups into which the incomes are divided and the available information pertaining to the income brackets (Gastwirth, 1972; Mehran, 1975; Cowell and Mehta, 1982; Giorgi and Pallini, 1987; Cowell, 1991). Table 1 summarizes the information requirements associated with the various upper bound proposals. It is apparent from the table that several of these proposals can be applied only if the group-mean income or the overall mean income is known. Therefore, if the mean income information is known, it seems reasonable to utilize this information in the computation of an upper bound of the Gini index. If there is absolutely no mean income information, or the mean income information is affected by substantial sampling or measurement errors, the upper bound proposals by Mehran (1975) and Silber (1990), which do not require mean income information, could be used. However, Mehran's upper bound is based on a relatively complicated GC-maximization problem that does not guarantee satisfaction of the relevant inequality constraints in practice (see Mehran, 1975, equation (2.5)). The problem with Silber's upper bound is that the derived coordinates of the points of intersection of the tangents to the LC at the observed points are plausible, though not mathematically precise; hence the resulting upper bound is not strictly precise.

The upper bound proposed by Ogwang (2003) is appealing since it can be implemented even with sparse mean income information. However, unlike Mehran or Silber's upper bound proposals, the methodology proposed by Ogwang cannot be directly applied if mean income information is not available. It would, therefore, be interesting to modify Ogwang's methodology in cases where there is absolutely no mean income information. The purpose of this paper, therefore, is to modify Ogwang's upper bound proposal in cases where there is no information on all the group mean incomes or the overall mean income but the limits of the income brackets are known. As will be seen below, the proposed methodology utilizes the fact that under certain conditions the GC-maximizing mean income for a given income bracket is equal to the geometric mean of the limits of that bracket. Thus, by substituting the (unknown) mean income for each income bracket with the corresponding value that maximizes GC, an upper bound of the Gini index can be obtained. It turns out that the proposed upper bound involves a very simple GC-maximization problem.

The format of the rest of the paper is as follows. In Section 2 an upper bound of the Gini index is derived, assuming that there is absolutely no mean income information but the limits of the income brackets are known. Illustrative examples showing the effects of changing the number and size of income brackets on the upper bound are provided in Section 3. Section 4 concludes.

2. An Upper Bound of the Gini Index in the Absence of Mean Income Information

It should be pointed out from the outset that the results in this section are extensions of some of the results in Ogwang's (2003) paper, wherein graphical illustrations of the bounds are provided. For the sake of brevity and clarity, the notations used in this paper are identical to those used in that paper.

Let us assume that the incomes of a particular set of income-receiving units (individuals, households) are divided into $k + 1$ income brackets with $a_0, a_1, \ldots, a_{k+1}$ as the interval endpoints ($0 \leq a_0 < a_1 < \ldots\ a_{k+1} \leq \infty$, which are provided. Let μ_i, μ, p_i, and $L(p_i)$ denote the mean income in the i-th income bracket (a_{i-1}, a_i), the overall mean income, the cumulative fraction of income receiving units with incomes less than a_i, and the corresponding cumulative fraction of income, respectively. The corresponding LC is defined by a set of ordered points $(p_i, L(p_i))$, which passes through the points (0,0) and (1,1). The diagonal line joining (0,0) and (1,1) is the perfect equality line (PEL). Note that the set-up of the income brackets precludes the possibility of negative incomes. The issue of the validity of negative incomes in distributional analysis is controversial. As is standard practice (e.g. Cowell, 1995), negative incomes could be ignored. Alternatively, they could be combined with the first income bracket $(0, a_1)$. Chen *et al.* (1982, 1985) and Berrebi and Silber (1985) have proposed ways of adjusting the Gini index when negative incomes are present, but the methods they propose are by no means perfect.

Since the Gini index is defined as twice the area between the LC and PEL, the lower bound of the Gini index is obtained by determining the area enclosed by the sequence of line segments joining the observed points on the LC and PEL and multiplying the result by two. The resulting estimate of the Gini index defines the lower bound since the piecewise-linear segments signify an equal distribution of incomes within each income bracket. Obtaining the upper bound entails determining the area formed by the tangents to the LC at all the observed points and PEL and multiplying the result by two. Forming the tangents yields the largest possible value of the Gini index, which is consistent with the observed points on the LC.

Ogwang (2003) showed that the lower bound of the Gini index, L, which is identical to Gastwirth's (1972) lower bound, is given by:

(1) $$L = 1 - \sum_{i=1}^{k+1} \beta_i^{-1} \left(L(p_i)^2 - L(p_{i-1})^2 \right)$$

where β_i is the slope of the line segment joining $(p_{i-1}, L(p_{i-1}))$ and $(p_i, L(p_i))$, which can be computed directly from the observed points on the LC. Since computing the lower bound of the Gini index does not require any information on the group mean incomes, the overall mean income, or the limits of the income brackets, no complications arise from the absence of such information. Therefore, we shall focus mainly on the upper bound.

Ogwang also showed that if the top income bracket is bounded from above (i.e. $a_{k+1} < \infty$), the corresponding upper bound of the Gini index is given by:

(2a) $$U = L + \sum_{i=1}^{k+1} (p_i - p_{i-1})^2 \left(\beta_i^* - \beta_i\right)\left(\beta_i - \beta_{i-1}^*\right)\left(\beta_i^* - \beta_{i-1}^*\right)^{-1}$$

where L is the lower bound given by equation (1) and β_i^* is the slope of the tangent to the LC at $(p_i, L(p_i))$.

If the top income bracket (a_k, a_{k+1}) is not bounded from above (i.e. $a_{k+1} = \infty$), the corresponding upper bound of the Gini index is given by:

(2b) $$U = L + \sum_{i=1}^{k} (p_i - p_{i-1})^2 \left(\beta_i^* - \beta_i\right)\left(\beta_i - \beta_{i-1}^*\right)\left(\beta_i^* - \beta_{i-1}^*\right)^{-1} + \left(\beta_{k+1} - \beta_k^*\right)(p_{k+1} - p_k)^2 .$$

In order to derive an upper bound of the Gini index in the absence of mean income information, it behooves us to rewrite equation (2a) in terms of the (known) limits of the income brackets and the overall mean income, μ, by substituting $\beta_i^* = a_i/\mu$ (Ogwang, 2003), which yields:

(3) $$U = L + \sum_{i=1}^{k+1} \mu^{-1} (p_i - p_{i-1})^2 (a_i - \mu_i)(\mu_i - a_{i-1})(a_i - a_{i-1})^{-1} .$$

It is also helpful to express the overall mean income, μ, in equation (3) in terms of the group mean incomes μ_i $i = 1,2, \ldots, k + 1$. Using the fact that $\mu_i = \beta_i\mu$ (Ogwang, 2003), or, equivalently, $\mu = \mu_i/\beta_i$, equation (3) can be rewritten in terms of the group mean incomes as:

(4) $$U = L + \sum_{i=1}^{k+1} \beta_i \frac{(p_i - p_{i-1})^2}{(a_i - a_{i-1})} \left[\frac{(a_i - \mu_i)(\mu_i - a_{i-1})}{\mu_i} \right].$$

From equation (4), it can be seen that the GC for income bracket (a_{i-1}, a_i) is given by:

(5) $$GC_i = \beta_i \frac{(p_i - p_{i-1})^2}{(a_i - a_{i-1})}\left[\frac{(a_i - \mu_i)(\mu_i - a_{i-1})}{\mu_i}\right].$$

For any given income bracket (a_{i-1}, a_i) $i = 1, 2, \ldots, k + 1$, the values of β_i, a_{i-1}, a_i, p_{i-1}, and p_i in equation (5) are known and, therefore, GC_i is a function of μ_i, the (unknown) mean income for that bracket. The task, therefore, boils down to determining the value of $\mu_i(a_{i-1} \leq \mu_i \leq a_i)$ that maximizes GC_i.

To determine the GC-maximizing value of μ_i, we differentiate the expression in the square brackets in equation (5) with respect to μ_i and equate to zero, i.e.:

(6) $$\frac{-\mu_i^2 + a_i a_{i-1}}{\mu_i^2} = 0.$$

Solving equation (6) for μ_i yields:

(7) $$\mu_i^* = \sqrt{a_i a_{i-1}}$$

which is the geometric mean of the limits of income bracket (a_{i-1}, a_i).

To show that μ_i^* is GC-maximizing, it suffices to establish that the second derivative of the expression in the square brackets in equation (5) with respect to μ_i, which is given by equation (8), is negative when evaluated at $\mu_i = \mu_i^*$:

(8) $$\frac{\{\mu_i^2[-2\mu_i]\} - \{\lfloor -\mu_i^2 + a_i a_{i-1} \rfloor 2\mu_i\}}{\mu_i^4} = \frac{-2\mu_i^3 + 2\mu_i^3 - 2\mu_i a_i a_{i-1}}{\mu_i^4} = \frac{-2a_i a_{i-1}}{\mu_i^3}$$

Clearly, the expression in equation (8) is negative when evaluated at $\mu_i = \mu_i^*$, provided that $a_{i-1} > 0$, which reveals a maximum point.

From equations (7) and (8), it can be deduced that if the incomes are divided into $k + 1$ income brackets with $a_0, a_1, \ldots, a_k$ as the interval endpoints, the GC for income bracket (a_{i-1}, a_i) is maximized when the (unknown) group mean income is equal to the geometric mean of the limits of that income bracket, provided that $a_{i-1} > 0$ and $a_i < \infty$. Once the GC-maximizing values of μ_i are determined for all the income brackets, the upper bound is obtained by first summing up the maximum possible GC values for all the income brackets then adding the result to the lower bound. Clearly, the proposed upper bound involves a very simple GC-maximization problem.

Minor complications arise when $a_0 = 0$ (i.e. the lowest income bracket is bounded from below by zero). Using the geometric approach, it can be shown that the GC for the first income bracket is given by:

(9) $$GC_1 = p_1^* L(p_1) = \frac{\left(\beta_1^* - \beta_1\right)}{\beta_1^*} p_1 L(p_1) = \left(1 - \frac{\beta_1}{\beta_1^*}\right) p_1 L(p_1)$$

where p_1^* is the point of intersection of the tangent to the LC at $(p_1, L(p_1))$ with the horizontal or p-axis (Ogwang, 2003, figure 1).

Since $\beta_1^* = a_1/\mu$, $\mu_1 = \beta_1\mu$ or, equivalently, $\mu = \mu_1/\beta_1$, and $L(p_1) = \beta_1 p_1$, equation (9) could be rewritten as:

$$GC_1 = \left(1 - \frac{\mu_1}{a_1}\right)\beta_1 p_1^2 \leq \beta_1 p_1^2. \tag{10}$$

The inequality relation in equation (10) holds since the (unknown) value of μ_1 cannot exceed a_1 in which case $0 \leq \mu_1/a_1 \leq 1$. Therefore, if the lowest income bracket, (a_0, a_1) is bounded from below by zero, the maximum possible value of GC_1 is $\beta_1 p_1^2$.

Minor complications also arise when $a_{k+1} = \infty$ (i.e. the highest income bracket is unbounded from above). From equation (2b), we note that the GC for the highest income bracket, (a_k, a_{k+1}), is given by:

$$GC_{k+1} = \left(\beta_{k+1} - \beta_k^*\right)(p_{k+1} - p_k)^2 \tag{11}$$

where $p_{k+1} = 1$ by construction.

Since $\beta_k^* = a_k/\mu$, and $\mu_{k+1} = \beta_{k+1}\mu$ (or, equivalently, $\mu = \mu_{k+1}/\beta_{k+1}$), equation (11) can be rewritten as:

$$GC_{k+1} = (\beta_{k+1} - a_k\beta_{k+1})(p_{k+1} - p_k)^2 \leq \beta_{k+1}(p_{k+1} - p_k)^2. \tag{12}$$

The inequality relation in equation (12) holds since the (unknown) value of μ_{k+1} cannot be less than a_k in which case $0 \leq a_k/\mu_{k+1} \leq 1$. Therefore, if the highest income bracket, (a_k, a_{k+1}), is not bounded from above, the maximum possible value of GC_{k+1} is $\beta_{k+1}(p_{k+1} - p_k)^2$.

In light of the above results, there are four possible scenarios that could arise in the determination of an upper bound of the Gini index in the absence of mean income information.

First, if $(0 < a_0 < a_1 < \ldots < a_{k+1} < \infty)$, the resulting upper bound is given by:

$$U = L + \sum_{i=1}^{k+1} \beta_i \frac{(p_i - p_{i-1})^2}{(a_i - a_{i-1})}\left[\frac{\left(a_i - \sqrt{(a_i a_{i-1})}\right)\left(\sqrt{(a_i a_{i-1})} - a_{i-1}\right)}{\sqrt{(a_i a_{i-1})}}\right]. \tag{13}$$

Second, if $(0 < a_0 < a_1 < \ldots < a_{k+1} = \infty)$, the resulting upper bound of the Gini index is given by:

$$U = L + \sum_{i=1}^{k} \beta_i \frac{(p_i - p_{i-1})^2}{(a_i - a_{i-1})}\left[\frac{\left(a_i - \sqrt{(a_i a_{i-1})}\right)\left(\sqrt{(a_i a_{i-1})} - a_{i-1}\right)}{\sqrt{(a_i a_{i-1})}}\right] + \beta_{k+1}(p_{k+1} - p_k)^2. \tag{14}$$

Third, if $(0 = a_0 < a_1 < \ldots < a_{k+1} < \infty)$, the resulting upper bound of the Gini index is given by:

TABLE 2
Distribution of Net Household Income

Income Bracket	Population Share	Income Share
Deciles		
294–582	0.1	0.026
582–827	0.1	0.042
827–1019	0.1	0.056
1019–1219	0.1	0.067
1219–1431	0.1	0.079
1431–1679	0.1	0.093
1679–1978	0.1	0.110
1978–2405	0.1	0.131
2405–2979	0.1	0.160
2979–4878	0.1	0.236
Quintiles		
294–827	0.2	0.068
827–1219	0.2	0.123
1219–1679	0.2	0.172
1679–2405	0.2	0.241
2405–4878	0.2	0.396

$$(15)\qquad U = L + \beta_1 p_1^2 + \sum_{i=2}^{k+1} \beta_i \frac{(p_i - p_{i-1})^2}{(a_i - a_{i-1})}\left[\frac{\left(a_i - \sqrt{(a_i a_{i-1})}\right)\left(\sqrt{(a_i a_{i-1})} - a_{i-1}\right)}{\sqrt{(a_i a_{i-1})}}\right].$$

Fourth, if ($0 = a_0 < a_1 < \ldots < a_{k+1} = \infty$), the resulting upper bound of the Gini index is given by:

$$(16)\qquad U = L + \beta_1 p_1^2 + \sum_{i=2}^{k} \beta_i \frac{(p_i - p_{i-1})^2}{(a_i - a_{i-1})}\left[\frac{\left(a_i - \sqrt{(a_i a_{i-1})}\right)\left(\sqrt{(a_i a_{i-1})} - a_{i-1}\right)}{\sqrt{(a_i a_{i-1})}}\right] + \beta_{k+1}(p_{k+1} - p_k)^2 .$$

In summary, the choice among equations (13) to (16) is dictated by the nature of the lowest and highest income brackets. Clearly, the availability of information on the limits of the income brackets tremendously simplifies the computation of an upper bound of the Gini index in the absence of mean income information.

3. Illustrative Examples: Effects of Changing the Magnitudes and Sizes of Income Brackets

To illustrate the proposed methodology, we used data on the distribution of net household income in Israel, originally derived from the Family Expenditure Survey 1986/87 reported by Fishelson (1994, appendix 3). For this data set, the minimum and maximum values for each decile are reported, but there is absolutely no mean income information, which provides an excellent test case for the methodology proposed in this paper. Table 2 provides details of these data (with minor adjustments to ensure that the upper bound of a particular income bracket coincides with the lower bound of the next higher income bracket). The bottom part of the table shows a re-categorization of the data by quintiles, which provides a

TABLE 3
ESTIMATED BOUNDS

Limits of First/Last Income Brackets		Quintiles		Deciles	
Lower/First	Upper/Last	Lower Bound	Upper Bound	Lower Bound	Upper Bound
294	4,878	0.3096	0.3363	0.3234	0.3303
Zero	Infinity	0.3096	0.4119	0.3234	0.3533
Zero	4,878	0.3096	0.3465	0.3234	0.3326
294	Infinity	0.3096	0.4017	0.3234	0.3511

stringent test case when the number of income brackets is very few. From the table, it can be seen that lower bound of the lowest income bracket is 294 and the upper bound of the highest income bracket is 4,878. However, we recalculated the bounds assuming that the lower bound of the lowest income bracket is zero and/or the upper bound of the highest income bracket is infinity.

The empirical results are reported in Table 3. Three points are apparent from the table. First, the results are very plausible. As expected, the bounds are dependent on the number and size of income brackets. Specifically, the bounds become narrower as the number of income brackets is increased (i.e. from quintiles to deciles). Second, using the highest income bracket with an infinite upper limit yields a larger bound than that with a finite upper limit, for any given characteristics of the other income brackets. Likewise, using zero as the lower bound of the smallest income bracket yields a larger bound than using a positive value for that income bracket, for any given characteristics of the other income brackets. Third, the estimated bounds are largest if the lower limit of the smallest income bracket is zero and the highest income bracket has an infinite upper limit. Accordingly, it seems sensible to set the lower bound of the lowest income bracket at zero if negative incomes are present, which ensures that the bounds are as large as possible within the data constraints. It should also be pointed out that the estimate of the Gini index computed by Fishelson, assuming a uniform distribution within each decile, is 0.327, which lies within the computed bounds in all cases.

4. Concluding Remarks

In this paper, an upper bound of the Gini index, based on grouped data, was proposed assuming that there is no information on all the group mean incomes as well as the overall mean income, but the limits of the income brackets are known. An important advantage of this proposal is that conventional formulas for the upper bound of the Gini index from LCs could be applied directly by substituting the (unknown) group mean income for each income bracket with the corresponding value that maximizes the GC for that income bracket. The proposed upper bound should ameliorate the sampling variability problem pointed out by McDonald and Ransom (1981), as it does not require information on mean income, an important source of sampling variability. Finally, our experience with data sets for four other countries (Australia, Italy, United Kingdom, and United States) indicates that the proposed method yields reasonably narrow bounds if there are 10 or more income brackets. Detailed results are reported in Ogwang (2006).

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