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WHEN DOES CHAINING REDUCE THE PAASCHE–LASPEYRES SPREAD? AN APPLICATION TO SCANNER DATA

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It is generally believed that chaining reduces the Paasche–Laspeyres spread if prices and quantities are monotonic over time. I consider three alternative definitions of monotonicity and show that none provide either necessary or sufficient conditions for chaining to reduce the Paasche–Laspeyres spread. What matters is the interaction between prices and quantities both in the same period and lagged one period. Sufficient conditions are derived, and the implications of these conditions for the measurement of inflation are considered. The paper concludes with an empirical illustration using scanner data.

1. INTRODUCTION

Paasche and Laspeyres are probably the two best known price index formulas. Both date back to the nineteenth century (see Diewert, 1993). A Laspeyres price index measures the change in the cost of buying the base period's basket of goods and services between the base and current period. Paasche by contrast measures the change in the cost of buying the current period's basket. Neither Paasche nor Laspeyres allow for the fact that, when prices change, consumers substitute away from products that have become relatively more expensive towards products that have become relatively cheaper. As a result, Laspeyres has an upward bias and Paasche a downward bias. It is argued in the literature therefore that superlative indexes should be used in preference to Paasche and Laspeyres (see Triplett, 1996). Superlative price indexes take account of the baskets of both the base and current periods, are free of substitution bias and approximate the cost-of-living index to the second order (see Diewert, 1978). There are, however, an infinite number of superlative formulas, the best known of which are Fisher, Walsh and Törnqvist. Even though the superlatives are free of substitution bias, this does not necessarily imply that the spread between them is smaller than the Paasche–Laspeyres spread (see Hill, 2006). Also, as Laspeyres and Paasche diverge from each other, the superlatives tend to do likewise. The Paasche–Laspeyres spread therefore is a useful indicator of the sensitivity of a price index comparison to the choice of formula, and hence is still of interest even if it is agreed that a superlative formula will be used to make the comparison.

Once a comparison is extended to cover three or more periods, a price index can have a fixed base or be chained.[1] The idea of chaining dates back at least to

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[1]Actually, a number of intermediate outcomes are also possible (see Hill, 2001).

Marshall (1887). A chained price index links together comparisons between adjacent time periods. For example, given annual data, a chained comparison between 2002 and 2005 is made indirectly by combining bilateral comparisons between 2002–03, 2003–04, and 2004–05. Chaining has been recommended in the literature for two main reasons. First, it allows the reference basket of goods and services to be updated as new goods appear on the market while other goods disappear and the relative importance of goods changes over time. Second, by constructing price indexes only between adjacent periods, chaining usually reduces the Paasche–Laspeyres spread and hence the sensitivity of the results to the choice of formula. This is because price and quantity patterns tend to be more similar for adjacent periods than for more distant periods. Chaining therefore may reduce the extent of the problem of index formula choice.

There are, however, some well documented exceptions to this general rule (see Forsyth and Fowler, 1981; Szulc, 1983). These authors find that chaining can increase the Paasche–Laspeyres spread if prices and quantities cycle or "bounce" (using Szulc's terminology) along the chain. Such systematic bouncing is unlikely in annual data unless there is a very pronounced business cycle. However, it can happen in quarterly or monthly data if seasonal goods are present. Reinsdorf (1998) shows how mean reversion in prices can also lead to bouncing. If the time interval between peaks and troughs of the cycles exactly matches the periods of the comparison, a situation akin to resonance can arise causing chaining to significantly increase the Paasche–Laspeyres spread.[2]

In recent years, the development of scanner data sets has added a new angle to this issue. Scanner data differ from the standard data previously used to construct price indexes in that they provide much greater detail and at higher frequency than was previously available. Previously, the quantity data typically had to be derived from consumer expenditure surveys that were infrequent and not very detailed. Scanner data by contrast provide the price of a particular product code in a particular supermarket in a particular week and the total number of units sold. The increasing availability of scanner data is forcing compilers of price indexes to start facing up to the problems created by sales. Sales can generate huge short-lived movements in prices and quantities on particular products. In traditional data sets, these movements are missed both because the quantity data are not available at the level of individual products, and because the index is only computed at a monthly or longer frequency.

Reinsdorf (1999) and Feenstra and Shapiro (2001) show that sales can have disastrous impacts on weekly chained price indexes (including superlative indexes). In fact, chained weekly Paasche–Laspeyres spreads computed on scanner data sets routinely exceed their direct counterparts, sometimes dramatically so. When the chained Paasche–Laspeyres spread rises, chained superlative indexes will also tend to diverge and drift erratically. These authors provide clear evidence of this. Feenstra and Shapiro (2001) go further and show that superlative indexes can even become biased in scanner data sets when quantities sold rise or fall in a systematic way towards the end of a sale.

[2]The problem of determining the optimal frequency of chaining is considered in Hill (2001) and Ehemann (2005).

It is important therefore even for users of superlative indexes to understand the impact of chaining on the Paasche–Laspeyres spread, and in particular the conditions under which chaining reduces and increases it. No clear consensus, however, has emerged in the literature on this point. As Hill (1988) observes:

> At the very least, it is necessary to know more about how the behavior of chain Laspeyres and Paasche indexes compares with that of their corresponding direct counterparts. It is sometimes assumed that chaining tends to reduce the index number spread between Laspeyres and Paasche indexes, and there are case studies to support this view. However, other studies suggest that chaining does not necessarily reduce the index number spread and may possibly even increase it. . . . There is little doubt that this uncertainty, indeed confusion, about the properties of chain indexes tends to discourage their use.

As was noted above, according to Forsyth and Fowler (1981) and Szulc (1983), chaining reduces the Paasche–Laspeyres spread if changes in prices and quantities are fairly monotonic over time. For example, Szulc (1983) states the problem as follows:

> Chain indexes may be considered superior to their direct counterparts when they provide a smooth passage between the base and the target time, rather than a detour.

This is undoubtedly a good rule of thumb. Deciding where to draw the line between a smooth path and a detour, however, is not entirely straightforward. The concept of smoothness seems to relate to some notion of monotonicity, and a detour to a violation of monotonicity. If so, Szulc's statement may not be entirely correct. I consider three alternative definitions of monotonicity in prices (quantities), and then show that none are either necessary or sufficient to ensure that chaining reduces the Paasche–Laspeyres spread. The problem with focusing on monotonicity is that it ignores the interaction between prices and quantities, which turns out to play a crucial role. I derive sufficient conditions for chaining to reduce (increase) the Paasche–Laspeyres spread which depend on the signs of correlation coefficients between price and quantity relatives both in the same period and lagged one period. The paper concludes with an empirical illustration using scanner data and a discussion of some of the implications of the findings.

2. Bilateral Price Indexes

Let $n = 1, \ldots, N$ index the basket of goods over which the price and quantity indexes are defined. The price of good n in time period t is denoted by p_{tn}, while the quantity of good n in period t is denoted by q_{tn}. It is assumed that $p_{tn}, q_{tn} > 0\ \forall n, t$. The expenditure share of good n in period t is denoted by s_{tn}.

$$s_{tn} = \frac{p_{tn} q_{tn}}{\sum_{i=1}^{N} p_{ti} q_{ti}} \tag{1}$$

Also, let P_{jk} and Q_{jk} denote bilateral price and quantity indexes between periods j and k. The Paasche, Laspeyres and Fisher price and quantity indexes are defined as follows:

(2) $$\text{Paasche:} \quad P_{jk}^{P} = \frac{\sum_{n=1}^{N} p_{kn} q_{kn}}{\sum_{n=1}^{N} p_{jn} q_{kn}} = \left\{ \sum_{n=1}^{N} \left(s_{kn} \frac{p_{jn}}{p_{kn}} \right) \right\}^{-1},$$

(3) $$Q_{jk}^{P} = \frac{\sum_{n=1}^{N} p_{kn} q_{kn}}{\sum_{n=1}^{N} p_{kn} q_{jn}} = \left\{ \sum_{i=1}^{N} \left(s_{kn} \frac{q_{jn}}{q_{kn}} \right) \right\}^{-1},$$

(4) $$\text{Laspeyres:} \quad P_{jk}^{L} = \frac{\sum_{n=1}^{N} p_{kn} q_{jn}}{\sum_{n=1}^{N} p_{jn} q_{jn}} = \sum_{n=1}^{N} \left(s_{jn} \frac{p_{kn}}{p_{jn}} \right),$$

(5) $$Q_{jk}^{L} = \frac{\sum_{n=1}^{N} p_{jn} q_{kn}}{\sum_{n=1}^{N} p_{jn} q_{jn}} = \sum_{n=1}^{N} \left(s_{jn} \frac{q_{kn}}{q_{jn}} \right),$$

(6) $$\text{Fisher:} \quad P_{jk}^{F} = \left(P_{jk}^{P} P_{jk}^{L} \right)^{1/2},$$

(7) $$Q_{jk}^{F} = \left(Q_{jk}^{P} Q_{jk}^{L} \right)^{1/2}.$$

Diewert (1978) argued that we should prefer price index formulas that are exact (i.e. equal to the cost of living index) for flexible expenditure functions (i.e. expenditure functions that are twice continuously differentiable and can approximate an arbitrary linearly-homogeneous function to the second order). He referred to price indexes that satisfy this condition as *superlative*. Diewert went on to identify a family of superlative formulas of the following form:

$$P_{jk}^{r} = \frac{\left(\sum_{n=1}^{N} s_{jn} \left(p_{kn} / p_{jn} \right)^{r/2} \right)^{1/r}}{\left(\sum_{n=1}^{N} s_{kn} \left(p_{kn} / p_{jn} \right)^{-r/2} \right)^{1/r}} \quad r \neq 0, \quad P_{jk}^{0} = \prod_{n=1}^{N} \left[\left(\frac{p_{kn}}{p_{jn}} \right)^{(s_{jn} + s_{kn})/2} \right],$$

where s_{jn} denotes the expenditure share of product n in time period t as defined in (1).

Although there are an infinite number of superlative price indexes, since the parameter r can take any finite positive or negative value, P_{jk}^{r} simplifies in an intuitively appealing manner for only three values of r. P_{jk}^{0} is the Törnqvist price index, P_{jk}^{1} is the implicit Walsh price index, and P_{jk}^{2} is the Fisher price index. It should be noted that neither Laspeyres nor Paasche is superlative. For most data sets, the Fisher, Walsh and Törnqvist indexes approximate each other closely. For example, for the case of the scanner data set used later in the paper, out of a total of 231 possible bilateral comparisons between different pairs of weeks, the maximum difference between Fisher, implicit Walsh and Törnqvist is a little under 1.1 percent (see Figure 1).

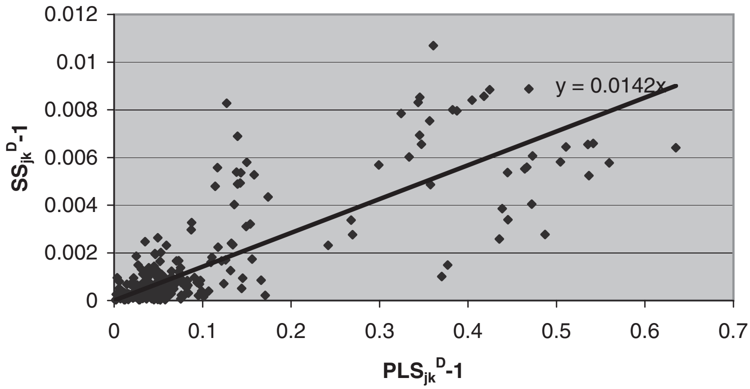

Figure 1. Scatter Plot of Paasche–Laspeyres Spreads Against Superlative Spreads

3. Paasche–Laspeyres Spreads

Although a clear consensus has emerged in the index-number literature that bilateral comparisons should be made using superlative formulas, the spread between a Paasche and Laspeyres index is still of interest since it is a useful indicator of the sensitivity of a bilateral comparison to the choice of index number formula.

To see this, consider the limiting cases where all formulas give the same answer. The data are consistent with the conditions for Hicks's (1946) aggregation theorem if $p_{kn} = \lambda p_{jn}$ for $n = 1, \ldots, N_{jk}$, where λ denotes a positive scalar. In this case, all acceptable price index formulas reduce to λ. The data are consistent with the conditions for Leontief's (1936) aggregation theorem if $q_{kn} = \mu q_{jn}$ for $n = 1, \ldots, N_{jk}$, where μ again is a positive scalar. In this case, all acceptable quantity index formulas reduce to μ. It follows, therefore, that all price index formulas should reduce to $\left(\sum_{n=1}^{N_{jk}} p_{kn}q_{kn}\right)\Big/\left(\mu\sum_{n=1}^{N_{jk}} p_{jn}q_{jn}\right)$, since price indexes can be obtained implicitly from quantity indexes.

The direct Paasche–Laspeyres spread is defined here as follows:[3]

$$PLS_{jk}^{D} = \frac{\max\left(P_{jk}^{P}, P_{jk}^{L}\right)}{\min\left(P_{jk}^{P}, P_{jk}^{L}\right)}. \tag{8}$$

It can be seen that $PLS_{jk}^{D} \geq 1$, $PLS_{jk}^{D} = PLS_{kj}^{D}$, and $PLS_{jj}^{D} = 1$. More generally, $PLS_{jk}^{D} = 1$ whenever the data satisfy the conditions for either Hicks's or Leontief's aggregation theorems.[4] This suggests that when PLS_{jk}^{D} is close to 1, a bilateral comparison will not be very sensitive to the choice of superlative formula.

Empirical support for this claim is provided in Figure 1. Let SS_{jk}^{D} denote the direct spread between Törnqvist, implicit Walsh and Fisher price indexes. That is:

[3]The ratio of Paasche to Laspeyres is the same for price and quantity indexes.

[4]The data satisfying the conditions for either Hicks's or Leontief's aggregation theorems is sufficient but not necessary for $PLS_{jk}^{D} = 0$. Diewert (2002) explores alternative sensitivity measures for which the data satisfying Hicks or Leontief aggregation are necessary and sufficient.

$$SS_{jk}^{D} = \frac{\max\left(P_{jk}^{0}, P_{jk}^{1}, P_{jk}^{2}\right)}{\min\left(P_{jk}^{0}, P_{jk}^{1}, P_{jk}^{2}\right)}.$$

Figure 1 plots $SS_{jk}^{D} - 1$ against $PLS_{jk}^{D} - 1$ for 231 bilateral comparisons between all possible combinations of 22 weeks of scanner data (see Section 6 for further details regarding this data set). The R^2 coefficient between $PLS_{jk}^{D} - 1$ and $SS_{jk}^{D} - 1$ is 0.6496. The ordinary-least-squares regression line (restricted to pass through the origin) is shown in Figure 1. The equation of this line is:

$$SS_{jk}^{D} - 1 = 0.0142\left(PLS_{jk}^{D} - 1\right).$$

That is, a 1 percent rise in $PLS_{jk}^{D} - 1$ acts to increase $SS_{jk}^{D} - 1$ on average by 0.0142 percent.

4. Sufficient Conditions for Determining the Impact of Chaining on the Paasche–Laspeyres Spread

Focusing on the case of a Laspeyres price index, a direct comparison between periods 1 and 3 is given by P_{13}^{L}, while a chained comparison between periods 1 and 3 links together a direct comparison between periods 1 and 2 with a direct comparison between periods 2 and 3, i.e. $P_{12}^{L}P_{23}^{L}$. A Laspeyres price index is intransitive. This means that, except in special cases, $P_{12}^{L}P_{23}^{L} \neq P_{13}$. The same is true for Paasche, and all superlative price (quantity) indexes.[5]

The chained Paasche–Laspeyres spread between periods 1 and 3 is defined below:

$$(9) \qquad PLS_{13}^{C} = \frac{\max\left(P_{12}^{P}P_{23}^{P}, P_{12}^{L}P_{23}^{L}\right)}{\min\left(P_{12}^{P}P_{23}^{P}, P_{12}^{L}P_{23}^{L}\right)}.$$

A crucial determinant of the relative magnitudes of PLS^{D} and PLS^{C} is the correlation between price relatives and quantity relatives. Here I distinguish between four such correlation coefficients.[6]

$$r_{pq,s2}^{23,21} = \frac{\sigma_{pq,s2}^{23,21}}{\left(\sigma_{p,s2}^{23}\right)\left(\sigma_{q,s2}^{21}\right)}, \quad r_{pq,s2}^{21,23} = \frac{\sigma_{pq,s2}^{21,23}}{\left(\sigma_{p,s2}^{21}\right)\left(\sigma_{q,s2}^{23}\right)}, \quad r_{pq,s1}^{12,12} = \frac{\sigma_{pq,s1}^{12,12}}{\left(\sigma_{p,s1}^{12}\right)\left(\sigma_{q,s1}^{12}\right)},$$

$$r_{pq,s2}^{23,23} = \frac{\sigma_{pq,s2}^{23,23}}{\left(\sigma_{p,s2}^{23}\right)\left(\sigma_{q,s2}^{23}\right)},$$

[5]It should be noted that if the data are consistent with the predictions of a behavioral model and the price index formula is exact for that model, then the chained index will match the direct index.

[6]Szulc (1983) and Reinsdorf (1998) also make use of correlation coefficients when examining the properties of chained indexes. They, however, focus on the correlation between different pairs of price (quantity) relatives (e.g. the ratio of the current period to previous period price versus the ratio of the current period to the base period price).

$$\text{where}\quad \sigma_{p,sj}^{jk}=\sqrt{\sum_{n=1}^{N}\left[s_{jn}\left(\frac{p_{kn}}{p_{jn}}-P_{jk}^{L}\right)^{2}\right]},$$

$$\sigma_{q,sj}^{jk}=\sqrt{\sum_{n=1}^{N}\left[s_{jn}\left(\frac{q_{kn}}{q_{jn}}-Q_{jk}^{L}\right)^{2}\right]},$$

$$\sigma_{pq,sj}^{jk,ji}=\sum_{n=1}^{N}\left[s_{jn}\left(\frac{p_{kn}}{p_{jn}}-P_{jk}^{L}\right)\left(\frac{q_{kn}}{q_{jn}}-Q_{ji}^{L}\right)\right].$$

The first two correlation coefficients determine the direction of chain drift in Laspeyres and Paasche price indexes.[7] The third and fourth determine which is larger out of Laspeyres and Paasche. These results are derived in the following three lemmas.

Lemma 1. $P_{12}^{L}P_{23}^{L}/P_{13}^{L}<1$ if and only if $r_{pq,s2}^{23,21}>0$.[8]

One must be careful interpreting this result. Positive correlation between p_{3n}/p_{2n} and q_{1n}/q_{2n} does not necessarily imply negative correlation between p_{3n}/p_{2n} and q_{2n}/q_{1n}. For example, the (unweighted) correlation between {1, 2, 1, 1, 2} and {0.5, 0.25, 4, 1, 8} is 0.3806, while the (unweighted) correlation between {1, 2, 1, 1, 2} and the reciprocal of {0.5, 0.25, 4, 1, 8} (i.e. {2, 4, 0.25, 1, 0.125}) is 0.3359.

Lemma 2. $P_{12}^{P}P_{23}^{P}/P_{13}^{P}>1$ if and only if $r_{pq,s2}^{21,23}>0$.

Lemma 3. $P_{12}^{L}/P_{12}^{P}>1$ if and only if $r_{pq,s1}^{12,12}<0$.[9]

Lemma 3 was first derived by von Bortkiewicz (1923). Von Bortkiewicz's contribution is discussed by, amongst others, Allen (1975), Kravis *et al.* (1982), and Szulc (1983). Since for most data sets Laspeyres exceeds Paasche, von Bortkiewicz's result implies that changes in prices and quantities are usually negatively correlated. This is consistent with the consumer substitution effect.

Lemmas 1, 2 and 3 can be combined to obtain a sufficient condition to ensure that chaining reduces the Paasche–Laspeyres spread.

Theorem 1. A sufficient condition to ensure that $PLS_{13}^{C}<PLS_{13}^{D}$ is that $r_{pq,s2}^{23,21}$ and $r_{pq,s2}^{21,23}$ have the same sign, which is opposite to the sign of $r_{pq,s1}^{12,12}$ and $r_{pq,s2}^{23,23}$.

Theorem 2. A sufficient condition to ensure that $PLS_{13}^{C}>PLS_{13}^{D}$ is that $r_{pq,s2}^{23,21}$, $r_{pq,s2}^{21,23}$, $r_{pq,s1}^{12,12}$ and $r_{pq,s2}^{23,23}$ all have the same sign.

5. Monotonic Prices and Quantities

It is generally assumed in the price index literature that chaining reduces the Paasche–Laspeyres spread when prices and quantities change smoothly over time, and conversely that it increases the Paasche–Laspeyres spread when prices and quantities fluctuate or bounce. These concepts, however, are not precisely defined.

[7]The chain drift of a Fisher index also depends on these correlation coefficients. A sufficient condition for $P_{12}^{F}P_{23}^{F}<P_{13}^{F}$ is that $r_{pq,s2}^{23,21}>0$ and $r_{pq,s2}^{21,23}<0$. Conversely, a sufficient condition for $P_{12}^{F}P_{23}^{F}>P_{13}^{F}$ is that $r_{pq,s2}^{23,21}<0$ and $r_{pq,s2}^{21,23}>0$.

[8]Proofs of lemmas and theorems are provided in the Appendix.

[9]Lemmas 1, 2 and 3 could equally well be expressed in terms of quantity indexes.

Smoothness relates to some notion of monotonicity, and bouncing to violations of monotonicity. It is not clear though what definition of monotonicity is most relevant in this context. Here I consider three alternative definitions.

Monotonicity in prices (quantities): Prices (quantities) are monotonic over periods 1, 2 and 3 if the following inequalities hold for each good n:

$$\frac{p_{2n}}{p_{1n}} > 1 \Rightarrow \frac{p_{3n}}{p_{2n}} > 1, \quad \frac{p_{2n}}{p_{1n}} < 1 \Rightarrow \frac{p_{3n}}{p_{2n}} < 1, \tag{10}$$

$$\frac{q_{2n}}{q_{1n}} > 1 \Rightarrow \frac{q_{3n}}{q_{2n}} > 1, \quad \frac{q_{2n}}{q_{1n}} < 1 \Rightarrow \frac{q_{3n}}{q_{2n}} < 1. \tag{11}$$

Monotonicity in median adjusted prices (quantities): Prices (quantities) are median adjusted monotonic over periods 1, 2 and 3 if the following inequalities hold for each good n:

$$\begin{aligned} &\left(\frac{p_{2n}}{p_{1n}}\right)\left(\frac{1}{P_{12}^M}\right) > 1 \Rightarrow \left(\frac{p_{3n}}{p_{2n}}\right)\left(\frac{1}{P_{23}^M}\right) > 1, \\ &\left(\frac{p_{2n}}{p_{1n}}\right)\left(\frac{1}{P_{12}^M}\right) < 1 \Rightarrow \left(\frac{p_{3n}}{p_{2n}}\right)\left(\frac{1}{P_{23}^M}\right) < 1, \end{aligned} \tag{12}$$

$$\begin{aligned} &\left(\frac{q_{2n}}{q_{1n}}\right)\left(\frac{1}{Q_{12}^M}\right) > 1 \Rightarrow \left(\frac{q_{3n}}{q_{2n}}\right)\left(\frac{1}{Q_{23}^M}\right) > 1, \\ &\left(\frac{q_{2n}}{q_{1n}}\right)\left(\frac{1}{Q_{12}^M}\right) < 1 \Rightarrow \left(\frac{q_{3n}}{q_{2n}}\right)\left(\frac{1}{Q_{23}^M}\right) < 1, \end{aligned} \tag{13}$$

where P_{st}^M denotes the median price relative and Q_{st}^M the median quantity relative in a comparison between periods s and t.

Monotonicity in mean adjusted prices (quantities): Prices (quantities) are mean adjusted monotonic over periods 1, 2 and 3 if the following inequalities hold for each good n:

$$\begin{aligned} &\left(\frac{p_{2n}}{p_{1n}}\right)\left(\frac{1}{P_{12}^F}\right) > 1 \Rightarrow \left(\frac{p_{3n}}{p_{2n}}\right)\left(\frac{1}{P_{23}^F}\right) > 1, \\ &\left(\frac{p_{2n}}{p_{1n}}\right)\left(\frac{1}{P_{12}^F}\right) < 1 \Rightarrow \left(\frac{p_{3n}}{p_{2n}}\right)\left(\frac{1}{P_{23}^F}\right) < 1, \end{aligned} \tag{14}$$

$$\begin{aligned} &\left(\frac{q_{2n}}{q_{1n}}\right)\left(\frac{1}{Q_{12}^F}\right) > 1 \Rightarrow \left(\frac{q_{3n}}{q_{2n}}\right)\left(\frac{1}{Q_{23}^F}\right) > 1, \\ &\left(\frac{q_{2n}}{q_{1n}}\right)\left(\frac{1}{Q_{12}^F}\right) < 1 \Rightarrow \left(\frac{q_{3n}}{q_{2n}}\right)\left(\frac{1}{Q_{23}^F}\right) < 1, \end{aligned} \tag{15}$$

where P_{st}^F and Q_{st}^F denote Fisher price and quantity indexes.

The first definition is of questionable relevance when examining the behavior of chained indexes. For example, it is entirely consistent with stagflation (i.e. a situation where all prices are rising and all quantities falling). Such a scenario does not in itself necessarily tell us anything about how a chained index will perform.[10] A notion of monotonicity that is more relevant in this context considers how prices (quantities) change relative to the average price (quantity) change. Given that goods with larger expenditure shares are of greater importance, it is preferable to focus on the weighted mean (i.e. the price index itself) as the reference average rather than the median. Hence the last of the three definitions of monotonicity probably best captures what the prevailing wisdom had in mind. That is, if a good rises in price more (less) than the mean in one period, then when the path is smooth we would expect it to do likewise in the next period. It turns out, however, that even monotonicity in mean adjusted prices and quantities by itself does not provide clear-cut results regarding the impact of chaining on the Paasche–Laspeyres spread.

Theorem 3. Monotonicity in prices and quantities, monotonicity in median adjusted prices and quantities, and monotonicity in mean adjusted prices and quantities are all neither necessary nor sufficient to ensure that chaining reduces the Paasche–Laspeyres spread.

The previous focus in the literature on monotonicity is misleading in that it ignores the interaction between prices and quantities. It implicitly assumes that $r^{12,12}_{pq,s1}$ and $r^{23,23}_{pq,s2}$ have the same sign, as do $r^{23,21}_{pq,s2}$ and $r^{21,23}_{pq,s2}$. An example helps illustrate the point. Suppose it is true for all goods that

$$(16)\qquad \frac{p_{2n}}{p_{1n}} > P^F_{12} \Rightarrow \frac{q_{2n}}{q_{1n}} < Q^F_{12}, \quad \frac{p_{2n}}{p_{1n}} < P^F_{12} \Rightarrow \frac{q_{2n}}{q_{1n}} > Q^F_{12},$$

$$(17)\qquad \frac{p_{3n}}{p_{2n}} > P^F_{23} \Rightarrow \frac{q_{3n}}{q_{2n}} < Q^F_{23}, \quad \frac{p_{3n}}{p_{2n}} < P^F_{23} \Rightarrow \frac{q_{3n}}{q_{2n}} > Q^F_{23}.$$

These inequalities are consistent with the consumer substitution effect (abstracting from income effects). Suppose also that prices and quantities satisfy mean adjusted monotonicity and hence the inequalities in (14) and (15). Combining (14), (15), (16) and (17), it follows that

$$(18)\qquad \frac{p_{1n}}{p_{2n}} > P^F_{21} \Rightarrow \frac{q_{3n}}{q_{2n}} > Q^F_{23}, \quad \frac{p_{1n}}{p_{2n}} < P^F_{21} \Rightarrow \frac{q_{3n}}{q_{2n}} < Q^F_{23},$$

$$(19)\qquad \frac{p_{3n}}{p_{2n}} > P^F_{23} \Rightarrow \frac{q_{1n}}{q_{2n}} > Q^F_{21}, \quad \frac{p_{3n}}{p_{2n}} < P^F_{23} \Rightarrow \frac{q_{1n}}{q_{2n}} < Q^F_{21}.$$

The consumer substitution effect acts to make both $r^{12,12}_{pq,s1}$ and $r^{23,23}_{pq,s2}$ negative, while the inequalities (18) and (19) tend to make both $r^{21,23}_{pq,s2}$ and $r^{23,21}_{pq,s2}$ positive, thus

[10]I thank an anonymous referee for pointing this out, and hence prompting me to develop alternative definitions of monotonicity.

satisfying the conditions of theorem 1. If instead the data are consistent with the producer substitution effect (i.e. all the inequalities involving quantities are reversed in (16) and (17)), while maintaining mean adjusted monotonicity, this will act to reverse the sign of all four correlation coefficients. The conditions for theorem 1 are again satisfied.

Suppose now instead that a consumer substitution effect is combined with cycles in mean-adjusted prices and quantities (i.e. all the inequalities involving comparisons between periods 2 and 3 are reversed in (14) and (15)). The following inequalities are obtained:

$$\text{(20)} \qquad \frac{p_{1n}}{p_{2n}} > P_{21}^F \Rightarrow \frac{q_{3n}}{q_{2n}} < Q_{23}^F, \quad \frac{p_{1n}}{p_{2n}} < P_{21}^F \Rightarrow \frac{q_{3n}}{q_{2n}} > Q_{23}^F,$$

$$\text{(21)} \qquad \frac{p_{3n}}{p_{2n}} > P_{23}^F \Rightarrow \frac{q_{1n}}{q_{2n}} < Q_{21}^F, \quad \frac{p_{3n}}{p_{2n}} < P_{23}^F \Rightarrow \frac{q_{1n}}{q_{2n}} > Q_{21}^F.$$

Now $r_{pq,s2}^{21,23}$ and $r_{pq,s2}^{23,21}$ will both tend to be negative, as by assumption are $r_{pq,s1}^{12,12}$ and $r_{pq,s2}^{23,23}$. Hence the conditions for theorem 2 are satisfied. If instead the data are consistent with the producer substitution effect, the sign of all four correlation coefficients will be reversed. The conditions for theorem 2 are again satisfied.

Once (16) and (17), or their producer substitution effect equivalents, cease to hold, however, the impact of monotonicity or cycles in mean adjusted prices and quantities becomes harder to discern. Counterintuitive results like those in examples 1 and 2 can arise (see the proof of theorem 3 in the Appendix). Indeed, $r_{pq,s2}^{23,21}$ and $r_{pq,s2}^{21,23}$ have opposite signs in both examples 1 and 2.[11] Either scenario implies that the sufficient conditions in theorems 1 and 2 are not satisfied, and places us in the twilight zone where outcomes are harder to predict.

6. The Impact of Chaining in Practice: The Case of Scanner Data

This section uses weekly scanner data from A. C. Nielsen.[12] The data set relates to a single supermarket in Australia over a 22 week period. It covers 68 coffee product codes that neither appeared nor disappeared during the period. I consider the impact of chaining on blocks of three consecutive periods. This yields a total of 20 observations (i.e. weeks 1–3, 2–4, 3–5, . . . , 20–22). Chaining reduces the Paasche–Laspeyres spread for 13 of the 20 observations, as shown in Table 1.

The corresponding correlation coefficients are shown in Table 2. If the first two correlation coefficients have one sign, while the second two have the opposite sign, then theorem 1 implies that chaining must reduce the Paasche–Laspeyres spread for that three-week sequence. Conversely, if all four have the same sign,

[11]The correlation coefficients for example 1 are $r_{pq}^{23,21} = -0.0098$; $r_{pq}^{21,23} = 0.1416$; $r_{pq}^{12,12} = -0.0755$; $r_{pq}^{23,23} = -0.0557$. The correlation coefficients for example 2 are $r_{pq}^{23,21} = -0.1290$; $r_{pq}^{21,23} = 0.0565$; $r_{pq}^{12,12} = -0.0480$; $r_{pq}^{23,23} = -0.0070$.

[12]A. C. Nielsen is one of the world's leading marketing information providers. It provides data and analysis of marketplace dynamics and consumer attitudes and behavior in over 100 countries.

TABLE 1

Price Indexes and Paasche–Laspeyres Spreads

Weeks	Direct Lasp	Direct Paasche	Direct Fisher	Chain Lasp	Chain Paasche	Chain Fisher	Direct PLS	Chain PLS
1–3	1.0021	0.8750	0.9364	0.9805	0.8625	0.9196	1.1452	1.1367
2–4	1.0297	0.9735	1.0012	1.0527	0.9159	0.9819	1.0577	1.1494*
3–5	1.1296	1.0171	1.0719	1.0904	1.0196	1.0544	1.1106	1.0695
4–6	1.0197	1.0117	1.0157	1.0190	1.0131	1.0160	1.0079	1.0058
5–7	0.9942	0.9920	0.9931	0.9920	0.9956	0.9938	1.0022	1.0036*
6–8	1.0167	1.0001	1.0084	1.0190	1.0063	1.0126	1.0165	1.0126
7–9	1.0144	0.9710	0.9925	1.0167	0.9687	0.9924	1.0447	1.0496*
8–10	1.0064	0.9748	0.9905	1.0145	0.9637	0.9887	1.0324	1.0527*
9–11	1.0232	0.9441	0.9829	1.0073	0.9479	0.9772	1.0838	1.0627
10–12	1.0004	0.9542	0.9770	1.0001	0.9266	0.9626	1.0485	1.0793*
11–13	1.0379	0.9515	0.9938	1.0210	0.9557	0.9878	1.0908	1.0684
12–14	1.0134	0.9675	0.9902	1.0104	0.9719	0.9909	1.0474	1.0397
13–15	1.0152	1.0011	1.0081	1.0178	1.0002	1.0090	1.0141	1.0176*
14–16	1.0425	0.6763	0.8396	1.0114	0.6788	0.8286	1.5415	1.4900
15–17	0.9899	0.7297	0.8499	0.9903	0.6629	0.8102	1.3565	1.4738*
16–18	1.0246	0.9571	0.9903	1.0178	0.9629	0.9900	1.0705	1.0570
17–19	1.2973	0.9986	1.1382	1.1181	0.9907	1.0525	1.2990	1.1285
18–20	1.1399	1.0275	1.0823	1.1249	1.0247	1.0736	1.1094	1.0979
19–21	1.0470	0.9965	1.0214	1.0450	0.9952	1.0198	1.0507	1.0501
20–22	1.0412	0.9911	1.0158	1.0259	0.9891	1.0074	1.0506	1.0372
1–22	1.0296	0.9838	1.0064	1.3155	0.5074	0.8170	1.0465	2.5927*

*Denotes cases where $PLS_{jk}^{D} < PLS_{jk}^{C}$.

TABLE 2

Correlation Coefficients

Weeks	$r_{pq}^{23,21}$	$r_{pq}^{21,23}$	$r_{pq}^{12,12}$	$r_{pq}^{23,23}$
1–3	0.3204	–0.1232	–0.7852	–0.7829
2–4	–0.3401	–0.3826	–0.7829	–0.6242
3–5	0.4357	0.0117	–0.6242	–0.2071
4–6	–0.0526	0.0266	–0.2071	0.1199
5–7	0.0454	0.1114	0.1199	–0.1340
6–8	–0.0875	0.2869	–0.1340	–0.3765
7–9	–0.0837	–0.0732	–0.3765	–0.7990
8–10	–0.2921	–0.1853	–0.7990	–0.5680
9–11	0.2851	0.2090	–0.5680	–0.8124
10–12	0.0091	–0.4113	–0.8124	–0.5468
11–13	0.3719	0.0585	–0.5468	–0.4983
12–14	0.1986	0.1406	–0.4983	–0.1810
13–15	0.1364	–0.0765	–0.1810	–0.5836
14–16	0.3854	0.0325	–0.5836	–0.9286
15–17	–0.0391	–0.3703	–0.9286	–0.6306
16–18	0.4791	0.0445	–0.6306	–0.6620
17–19	0.8715	–0.1189	–0.6620	–0.5144
18–20	0.2204	–0.0533	–0.5144	–0.2967
19–21	0.0713	–0.0370	–0.2967	–0.6761
20–22	0.5987	–0.0343	–0.6761	0.1378

then theorem 2 implies that chaining must increase the Paasche–Laspeyres spread. Six observations (weeks 3–5, 9–11, 11–13, 12–14, 14–16, 16–18) satisfy the sufficient conditions in theorem 1. Five observations (weeks 2–4, 7–9, 8–10, 13–15, 15–17) satisfy the sufficient conditions in theorem 2. The sufficient conditions of either theorems 1 or 2 are therefore satisfied by 11 out of 20 observations. This gives some indication of how often they can be applied to real-world data.

The last row in Table 1 shows the impact of chaining over the full 22 week period. The results confirm the findings of Reinsdorf (1999) and Feenstra and Shapiro (2001). According to a weekly chained Laspeyres price index, prices rose by 31 percent over the 22 week period, while according to Paasche, prices fell by 50 percent! There is also clear evidence of drift in the weekly chained Fisher price index, which fell by nearly 20 percent. By contrast, according to the direct Fisher price index, prices rose by 0.6 percent.

Theorems 1 and 2 shed new light on the impact of a one-week sale on the chained Paasche–Laspeyres spread. The correlation coefficients $r^{12,12}_{pq,s1}$ and $r^{23,23}_{pq,s2}$ should normally both be negative in a consumer data set, due to the substitution effect. This is the case for 17 of the 20 observations in Table 2. Of the remaining three observations, in each case one of the coefficients is positive while the other is negative. By strengthening the substitution effect, a sale should act to make the correlation coefficients $r^{12,12}_{pq,s1}$ and $r^{23,23}_{pq,s2}$ even more negative than they would be otherwise. The problem for chaining arises as a result of the impact of sales on the lagged correlation coefficients $r^{21,23}_{pq,s2}$ and $r^{23,21}_{pq,s2}$. In normal circumstances these should both be positive. However, a one-week sale with a large associated substitution effect can cause the sign of one or both of these correlation coefficients to change.

A good example of this is the observation for weeks 2–4, which satisfies the sufficient conditions in theorem 2. The poor performance of chaining in this case is largely attributable to a sale on product 19. If product 19 is removed from the data set, PLS^C falls from 1.1494 to 1.0488, while PLS^D falls from 1.0577 to 1.0487. The price sequence for product 19 over weeks 2–5 was \$6.48, \$4.91, \$5.53, \$6.48. The corresponding quantity sequence is 28, 319, 65, 16. To place the huge rise in quantity purchased in week 3 in perspective, the next highest quantity purchased across the other 67 coffee products in this week was 47. The impact of the sale was to create an extremely small value of q_{1n}/q_{2n} with a large associated value of p_{3n}/p_{2n} for product 19, and an extremely small value of q_{3n}/q_{2n} with a large associated value of p_{1n}/p_{2n}. This could cause a change in the sign of one or both of $r^{23,21}_{pq,s_2}$ and $r^{21,23}_{pq,s_2}$. This indeed is the case for $r^{23,21}_{pq,s_2}$ which equals –0.3401 when product 19 is included, and 0.0289 when it is excluded. In other words, the sale on product 19 single handedly changes the sign of $r^{23,21}_{pq,s_2}$ from positive to negative. In this case, product 19 does not change the sign of $r^{21,23}_{pq,s_2}$ which is negative either way. It equals –0.3826 when product 19 is included, and –0.0554 when it is excluded. The fact that $r^{21,23}_{pq,s_2}$ is negative even in the absence of product 19 may be attributable to concurrent, albeit less pronounced, sales on other products in the sample. Removal of these products from the sample as well would presumably change the sign of $r^{21,23}_{pq,s_2}$ from negative to positive. As expected, the inclusion of product 19 strengthens the substitution effect and hence the negative correlation between price and quantity relatives in the same period. $r^{12,12}_{pq,s_1}$ equal –0.7829 when product 19 is included and –0.2188 when it is excluded, and $r^{23,23}_{pq,s_1}$ equal –0.6242 when product 19 is included, and –0.5390 when it is excluded. Hence it can be seen how a sale on a single product in the sample can cause the data to switch from satisfying the sufficient conditions of theorem 1 to satisfying the sufficient conditions of theorem 2.[13]

Inspection of Table 1 reveals another interesting phenomenon. Comparing the direct and chained Paasche–Laspeyres spreads over three week periods (i.e. the last two columns of Table 1) shows that on average they are of similar size. The geometric mean of the three period PLS^D is 1.102, while for PLS^C it is 1.097. And yet over the 22 week period, the difference is huge as shown in the final row of Table 1.[14] The PLS^C compound while PLS^D do not as the number of periods in the comparison rise. Why is this the case? The PLS^C compound whenever there is a clear consumer (producer) substitution effect, as is the case in the scanner data set, since chained Laspeyres exceeds (is less than) Paasche over any three period sample. If instead no systematic relationship existed between the prices and quantities, then on average chained Laspeyres would exceed chained Paasche only half the time and the three period PLS^C for periods 1–3, 3–5, 5–7, etc, would offset each other in the 22 period PLS^C rather than compounding. There would be no reason to expect the 22 period PLS^C to be any larger than the 3 period PLS^C. The fact that the 22 period PLS^D is not larger than the 3 period PLS^D is because movements in the data are predominantly short term, and there is no particular long run trend (which is not that surprising since the 22 periods still amount to less than half a year). Hence there is no reason to expect periods 1 and 3 to be any more similar than periods 1 and 22 or therefore $PLS^D_{1,3}$ to be any smaller than $PLS^D_{1,22}$. The same patterns are likely to be observed in other weekly scanner data sets.

7. Conclusion

Superlative (and most other) index number formulas tend to diverge from each other as the Paasche–Laspeyres spread rises. Whether or not chaining reduces the Paasche–Laspeyres spread and hence the sensitivity of price indexes to the choice of formula depends on the correlation between price and quantity relatives both in the same period and lagged one period. Monotonic prices and quantities do not, in general, guarantee that chaining will reduce the Paasche–Laspeyres spread. The sufficient conditions derived in the paper also help shed light on the often erratic behavior of chained price indexes in scanner data sets.

Appendix

Proof of Lemma 1.[15] First it is useful to express the ratio $P^L_{12}P^L_{23}/P^L_{13}$ as follows:

[13]Admittedly, the example considered here does not quite achieve this since $r^{21,23}_{pq,s_2}$ is still just negative even in the absence of product 19.

[14]I would like to thank the other anonymous referee for drawing this issue to my attention.

[15]This proof is a generalization of the approach used by Frisch (1936) to derive similar conditions for the Sauerbeck (or Carli) index, and by Allen (1975, p. 186) for the Laspeyres index. It should be noted that the arguments of the correlation coefficients and their weights differ in Allen's analysis from those derived here.

(22)
$$\frac{P_{12}^L P_{23}^L}{P_{13}^L} = \frac{\sum_{n=1}^{N}\left(s_{1n}\frac{p_{2n}}{p_{1n}}\right)\sum_{n=1}^{N}\left(s_{2n}\frac{p_{3n}}{p_{2n}}\right)}{\sum_{n=1}^{N}\left(s_{1n}\frac{p_{3n}}{p_{1n}}\right)}.$$

Rearrangement of (22) yields the following expression:

(23)
$$\frac{P_{12}^L P_{23}^L}{P_{13}^L} = \frac{\sum_{n=1}^{N}\left(s_{2n}\frac{q_{1n}}{q_{2n}}\right)\sum_{n=1}^{N}\left(s_{2n}\frac{p_{3n}}{p_{2n}}\right)}{\sum_{n=1}^{N}\left(s_{2n}\frac{q_{1n}}{q_{2n}}\frac{p_{3n}}{p_{2n}}\right)}.$$

Now define the following terms:

$$\begin{gathered}
\bar{x} = \sum_{n=1}^{N}\alpha_n x_n, \quad \bar{y} = \sum_{n=1}^{N}\alpha_n y_n, \quad \sum_{n=1}^{N}\alpha_n = 1, \\
\sigma_x^2 = \sum_{n=1}^{N}\left\{\alpha_n(x_n-\bar{x})^2\right\}, \quad \sigma_y^2 = \sum_{n=1}^{N}\left\{\alpha_n(y_n-\bar{y})^2\right\}, \\
\sigma_{xy} = \sum_{n=1}^{N}\{\alpha_n(x_n-\bar{x})(y_n-\bar{y})\}, \quad r_{xy} = \frac{\sigma_{xy}}{\sigma_x\sigma_y}.
\end{gathered}$$

Rewriting the formula for the correlation coefficient r_{xy}, the following expression is obtained:

(24)
$$\sum_{n=1}^{N}\alpha_n x_n y_n = \bar{x}\bar{y} + \sigma_x\sigma_y r_{xy}.$$

Now let

(25)
$$x_n = \frac{q_{1n}}{q_{2n}}, \quad y_n = \frac{p_{3n}}{p_{2n}}, \quad \alpha_n = s_{2n}.$$

Substituting (25) into (24), the following expression is obtained:

(26)
$$\sum_{n=1}^{N}\left(s_{2n}\frac{q_{1n}}{q_{2n}}\frac{p_{3n}}{p_{2n}}\right) = \bar{x}\bar{y} + \sigma_x\sigma_y r_{xy},$$

where

$$\bar{x} = \sum_{n=1}^{N}\left(\alpha_n\frac{q_{1n}}{q_{2n}}\right) = Q_{21}^L \quad \text{and} \quad \bar{y} = \sum_{n=1}^{N}\left(\alpha_n\frac{p_{3n}}{p_{2n}}\right) = P_{23}^L.$$

Substitution of (26) into (23) yields the following expression:

$$\frac{P_{12}^{L}P_{23}^{L}}{P_{13}^{L}}=\frac{\overline{x}\overline{y}}{\overline{x}\overline{y}+\sigma_x\sigma_y r_{xy}}=\frac{1}{1+\dfrac{\sigma_x\sigma_y r_{xy}}{\overline{x}\overline{y}}}. \tag{27}$$

Since s_x, s_y, $\bar{x}$ and $\bar{y}$ are all positive, (27) therefore implies that $P_{13}^{L}>P_{12}^{L}P_{23}^{L}$ if $r_{xy}>0$, where r_{xy} is the weighted correlation coefficient between p_{3n}/p_{2n} and q_{1n}/q_{2n}, with the weight on good n equal to s_{2n}. □

Lemmas 2 and 3 are proved in a similar way.

Proof of Theorem 1. Suppose that the first two correlation coefficients are positive and the last two are negative. It follows from lemma 3 that $P_{12}^{L}>P_{12}^{P}$ and $P_{23}^{L}>P_{23}^{P}$.

$$\Rightarrow PLS_{13}^{D}=\frac{P_{13}^{L}}{P_{13}^{P}} \quad PLS_{13}^{C}=\frac{P_{12}^{L}P_{23}^{L}}{P_{12}^{P}P_{23}^{P}}$$

It follows from lemma 1 that $P_{12}^{L}P_{23}^{L}<P_{13}^{L}$ and from lemma 2 that $P_{12}^{P}P_{23}^{P}>P_{13}^{P}$. Combining these results:

$$PLS_{13}^{C}=\frac{P_{12}^{L}P_{23}^{L}}{P_{12}^{P}P_{23}^{P}}<\frac{P_{13}^{L}}{P_{13}^{P}}=PLS_{13}^{D}.$$

Suppose now instead that the first two correlation coefficients are negative and the last two are positive. It follows from lemma 3 that $P_{12}^{L}<P_{12}^{P}$ and $P_{23}^{L}<P_{23}^{P}$.

$$\Rightarrow PLS_{13}^{D}=\frac{P_{13}^{P}}{P_{13}^{L}} \quad PLS_{13}^{C}=\frac{P_{12}^{P}P_{23}^{P}}{P_{12}^{L}P_{23}^{L}}$$

It follows from lemma 1 that $P_{12}^{L}P_{23}^{L}>P_{13}^{L}$ and from lemma 2 that $P_{12}^{P}P_{23}^{P}<P_{13}^{P}$. Combining these results:

$$PLS_{13}^{C}=\frac{P_{12}^{P}P_{23}^{P}}{P_{12}^{L}P_{23}^{L}}<\frac{P_{13}^{P}}{P_{13}^{L}}=PLS_{13}^{D}. \qquad \square$$

The proof of Theorem 2 is a slight variant of the above.

Proof of Theorem 3. That none of the monotonicity conditions are sufficient to ensure that chaining reduces the Paasche–Laspeyres spread is demonstrated by Example 1.

Example 1	Period 1	Period 2	Period 3	Period 1	Period 2	Period 3
Good 1	$p_{11}=0.21$	$p_{21}=0.19$	$p_{31}=0.18$	$q_{11}=1.00$	$q_{21}=1.01$	$q_{31}=1.09$
Good 2	$p_{12}=1.00$	$p_{22}=1.01$	$p_{32}=2.90$	$q_{12}=1.00$	$q_{22}=1.76$	$q_{32}=2.00$
Good 3	$p_{13}=1.00$	$p_{23}=1.02$	$p_{33}=3.00$	$q_{13}=1.40$	$q_{23}=1.01$	$q_{33}=1.00$

Prices and quantities over the three time periods satisfy all three monotonicity conditions. For the case of median adjusted monotonicity, product 2 is the median for prices, while product 1 is the median for quantities. For mean adjusted monotonicity, the relevant Fisher indexes are $P_{12}^F = 1.0064$; $P_{23}^F = 2.7714$; $Q_{12}^F = 1.1420$; $Q_{23}^F = 1.0821$. Chaining nevertheless increases the Paasche–Laspeyres spread.

$$\frac{P_{12}^L}{P_{12}^P} = 1.000989 \quad \frac{P_{23}^L}{P_{23}^P} = 1.000605 \quad \frac{P_{13}^L}{P_{13}^P} = 1.001121$$

$$\Rightarrow 1.001595 = \frac{P_{12}^L}{P_{12}^P}\frac{P_{23}^L}{P_{23}^P} = PLS_{13}^C > PLS_{13}^D = \frac{P_{13}^L}{P_{13}^P} = 1.001121$$

That none of the monotonicity conditions are necessary to ensure that chaining reduces the Paasche–Laspeyres spread is demonstrated by Example 2.

Example 2	Period 1	Period 2	Period 3	Period 1	Period 2	Period 3
Good 1	$p_{11} = 4.00$	$p_{21} = 5.60$	$p_{31} = 5.00$	$q_{11} = 1.50$	$q_{21} = 1.20$	$q_{31} = 1.33$
Good 2	$p_{12} = 7.80$	$p_{22} = 8.00$	$p_{32} = 7.30$	$q_{12} = 1.30$	$q_{22} = 1.00$	$q_{32} = 1.20$
Good 3	$p_{13} = 1.00$	$p_{23} = 1.10$	$p_{33} = 1.00$	$q_{13} = 1.95$	$q_{23} = 2.80$	$q_{33} = 1.90$

In this example, all three monotonicity conditions are violated. The prices and quantities of all three products fluctuate over the three periods, thus ensuring that monotonicity in prices and quantities are violated. Product 3 provides the median price change in both the period 1–2 and 2–3 comparisons. Product 1 rises in price more than the median from period 1 to 2, and less than the median from period 2 to 3. For product 1 the pattern is reversed. Product 1 provides the median quantity change in both the period 1–2 and 2–3 comparisons. Product 3 rises in quantity more than the median from period 1 to 2, and less than the median from period 2 to 3. For product 2 the pattern is reversed. All three products violate mean adjusted monotonicity in both prices and quantities. Product 1 rises more in price than P_{12}^F from period 1 to 2 and less than P_{23}^F from period 2 to 3. For products 2 and 3 the pattern is reversed. Product 3 rises more in quantity than Q_{12}^F from period 1 to 2 and less than Q_{23}^F from period 2 to 3. For products 1 and 2 the pattern is reversed. (Note: the relevant Fisher indexes are $P_{12}^F = 1.4788$; $P_{23}^F = 0.4116$; $Q_{12}^F = 0.9406$; $Q_{23}^F = 1.7079$.)

Chaining, nevertheless, reduces the Paasche–Laspeyres spread in example 2.

$$\frac{P_{12}^L}{P_{12}^P} = 1.00171 \quad \frac{P_{23}^L}{P_{23}^P} = 1.00001 \quad \frac{P_{13}^L}{P_{13}^P} = 1.00283$$

$$\Rightarrow 1.00172 = \frac{P_{12}^L}{P_{12}^P}\frac{P_{23}^L}{P_{23}^P} = PLS_{13}^C < PLS_{13}^D = \frac{P_{13}^L}{P_{13}^P} = 1.00283 \qquad \square$$

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