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GDP GROWTH ACCOUNTING: A NATIONAL INCOME FUNCTION APPROACH

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In this paper, we provide a decomposition of GDP growth that is complete and exact for the translog national income function and for the Törnqvist index of real GDP. The contributions of changes in output quantities, factor prices and total factor productivity are identified. Special consideration is given to foreign trade, with imports treated as a negative output. Annual estimates for the United States are reported for the period 1948–98.

1. INTRODUCTION

Over the past 50 years, U.S. gross domestic product (GDP) has been multiplied by a factor of 30.[1] Obviously, this remarkable achievement has real as well as nominal causes. On the one hand, it reflects a massive increase in the production of goods and services, made possible, in part, by technological progress, and leading to a steadily increasing demand for domestic primary factors. On the other hand, it is also indicative of rising factor prices. It is of interest to identify the contributions of these various effects in explaining the changes in GDP. Indeed, the Bureau of Economic Analysis (BEA) routinely provides a decomposition of real GDP growth within its system of national accounts,[2] and the statistical agencies of many other countries do likewise.

Until recently U.S. real GDP figures were computed as runs of direct Laspeyres quantity indexes. This is still the case in most countries around the world. Deriving an additive decomposition of real GDP growth that is exact for the Laspeyres functional form is straightforward.[3] However, it is well known that Laspeyres indexes have very restrictive properties, and they can only provide a first-order approximation to an arbitrary aggregator function. In 1996, the BEA started to publish real GDP figures using the almost ideal Fisher quantity index. Moreover, it switched to chain-type indexes. The Fisher index is superlative in the sense of Diewert (1976) in that it is exact for a functional form (the square-rooted quadratic) that can provide a second-order approximation to an arbitrary aggregator

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[1]In what follows, the term GDP on its own is used to designate *nominal* GDP. When referring to the corresponding quantity index, we will always use the term *real* GDP.

[2]See the National Income and Product Accounts (NIPA) Table 8.2, in particular.

[3]See footnote 14 for details.

function. The BEA then had to find a new way to provide a decomposition of real GDP growth. In a recent paper, Reinsdorf, Diewert, and Ehemann (2000) have proposed three different methods to achieve this, including the method currently used by the BEA.[4] Each one can be justified on different grounds. They are not numerically equivalent, although they approximate each other to the second order. None of them is very intuitive, however. Moreover, the square-rooted quadratic functional form, which they seek to replicate, does not seem to be well suited to represent a GDP or a national income function, except in the restrictive case where the outputs are globally separable from the fixed inputs, or, what amounts to essentially the same, when the number of outputs or the number of primary inputs is equal to one.[5] Note also that the BEA decomposition is for real GDP only, i.e. price and productivity effects are left out.

Recently, a number of authors have decomposed GDP growth by emphasizing the role of factor endowments, technology and output prices. Drawing on the pioneering work of Diewert and Morrison (1986), Kohli (1990) has proposed a decomposition of GDP that is both complete and exact for the translog GDP function.[6] The approach proposed in this paper extends this earlier work by focusing on the dual price and quantity variables. It emphasizes the demand for goods, the structure of foreign trade, the influence of factor rental prices, and the role of technological change. This approach is particularly relevant if one views output as being largely demand determined and factor prices as rigid in the very short run. It is also directly comparable to the growth decomposition as it is typically undertaken by statistical agencies, except that it goes further by incorporating price effects and technological change. Being based on the national income function approach to modeling the production sector of an open economy,[7] it rests on a solid theoretical foundation. Furthermore, being exact for the translog functional form, it is superlative in the sense of Diewert (1976).

2. Description of the Aggregate Technology

Assume that aggregate production involves J primary inputs and I final goods. We denote the vector of primary input quantities at time t by $\mathbf{x}_t \equiv [x_{j,t}]$, $j = 1, \ldots, J$ and the vector of output quantities by $\mathbf{y}_t \equiv [y_{i,t}]$, $i = 1, \ldots, I$. The corresponding prices are $\mathbf{w}_t \equiv [w_{j,t}]$, $j = 1, \ldots, J$ and $\mathbf{p}_t \equiv [p_{i,t}]$, $i = 1, \ldots, I$, respectively. Quantity $y_{i,t}$ is positive if component i is a net output, and it is negative if it is a net input. In what follows, we will treat imports as a negative output. This is consistent with the treatment of imports in the national accounts, and it recognizes the fact that most, if not all, imports are "middle products," to use the terminology of Sanyal and Jones (1982).

Let T_t be the production possibilities set at time t. We assume free disposals, constant returns to scale, and convexity. Under competitive conditions, the aggregate technology can also be represented by the national income function which is

[4]See footnote 14 for the precise formula.
[5]This question has been investigated in greater detail by Diewert (1992).
[6]See Kohli (1978) and Woodland (1982) for a derivation of the GDP function.
[7]See Kohli (1991).

the solution to minimizing the total cost of production, for given output quantities, given factor rental prices, and the current state of the technology[8]:

$$C(\mathbf{y}_t, \mathbf{w}_t, t) \equiv \min_{\mathbf{x}_t} \left\{ \sum_j w_{j,t} x_{j,t} : (\mathbf{y}_t, \mathbf{x}_t) \in T_t \right\}. \tag{1}$$

Given the assumptions made above, the national income function is linearly homogeneous and convex in output (including import) quantities; it is nondecreasing in the quantities of net outputs and nonincreasing in the quantities of net inputs; it is linearly homogeneous, concave, and nondecreasing in domestic factor rental prices.[9]

Under competitive conditions, the marginal cost of each output is equal to its price. Differentiation of the national income function with respect to the fixed quantities thus yields the output price—or inverse output supply—functions:

$$p_{i,t}(\mathbf{y}_t, \mathbf{w}_t, t) = \pm \frac{\partial C(\mathbf{y}_t, \mathbf{w}_t, t)}{\partial y_{i,t}}, \quad i = 1, \ldots, I, \tag{2}$$

where the sign is negative for net inputs, and positive otherwise. Furthermore, Shephard's (1953) Lemma implies that the differentiation of $C(\cdot)$ with respect to the factor rental prices yields the cost minimizing demand for primary inputs:

$$x_{j,t}(\mathbf{y}_t, \mathbf{w}_t, t) = \frac{\partial C(\mathbf{y}_t, \mathbf{w}_t, t)}{\partial w_{j,t}}, \quad j = 1, \ldots, J. \tag{3}$$

The translog functional form is well suited to represent the national income function. It is as follows[10]:

$$\begin{aligned} \ln C(\cdot) = \alpha_0 &+ \sum \alpha_i \ln y_{i,t} + \sum \beta_j \ln w_{j,t} + \frac{1}{2} \sum \sum \gamma_{ih} \ln y_{i,t} \ln y_{h,t} \\ &+ \frac{1}{2} \sum \sum \phi_{jk} \ln w_{j,t} \ln w_{k,t} + \sum \sum \delta_{ij} \ln y_{i,t} \ln w_{j,t} \\ &+ \sum \delta_{iT} \ln y_{i,t} t + \sum \phi_{jT} \ln w_{j,t} t + \beta_T t + \frac{1}{2} \phi_{TT} t^2, \\ & i,h = 1, \ldots I; j,k = 1, \ldots, J, \end{aligned} \tag{4}$$

where $\Sigma\alpha_i = 1$, $\Sigma\beta_j = 1$, $\gamma_{ih} = \gamma_{hi}$, $\phi_{jk} = \phi_{kj}$, $\Sigma\gamma_{ih} = 0$, $\Sigma\phi_{jk} = 0$, $\Sigma_i\delta_{ij} = 0$, $\Sigma_j\delta_{ij} = 0$, $\Sigma\delta_{iT} = 0$, and $\Sigma\phi_{jT} = 0$. National income function (4) is globally separable between outputs, on one hand, and inputs and time, on the other hand, if and only if $\delta_{ij} = \delta_{iT} = 0$, $\forall i, j$.

In the translog case, it is most convenient to derive the inverse output supply and input demand functions (2)–(3) in share form:

$$s_{i,t} = \alpha_i + \sum \gamma_{ih} \ln y_{h,t} + \sum \delta_{ij} \ln w_{j,t} + \delta_{iT} t, \quad i = 1, \ldots, I \tag{5}$$

[8]The national income function has all the properties of a joint cost function, and it is the dual of the GDP function; see Hall (1973), McFadden (1978), Woodland (1982) and Kohli (1991) for details.

[9]See McFadden (1978), for instance. The national income function is globally separable between outputs, on the one hand, and inputs and time, on the other hand, if it can be written as: $C(\mathbf{y}_t, \mathbf{w}_t, t) = h(\mathbf{y}_t)c(\mathbf{w}_t, t)$, where $h(\mathbf{y}_t)$ is a factor requirements function and $c(\mathbf{w}_t, t)$ is a unit cost function.

[10]See Christensen, Jorgenson, and Lau (1973) and Diewert (1974).

(6) $$s_{j,t} = \beta_j + \sum \delta_{ij} \ln y_{i,t} + \sum \phi_{jk} \ln w_{k,t} + \phi_{jT} t, \quad j = 1, \ldots, J,$$

where $s_{i,t} \equiv p_{i,t} y_{i,t} / C(\cdot)$ and $s_{j,t} \equiv w_j x_j / C(\cdot)$ are the GDP or national income shares of output i and factor j, respectively.

Furthermore, let $s_{T,t}$ be the instantaneous rate of technological change at time t, i.e. $s_{T,t} \equiv \partial \ln C(\mathbf{y}_t, \mathbf{w}_t, t)/\partial t$. It then follows from (4) that:

(7) $$s_{T,t} = \beta_T + \sum \delta_{iT} \ln y_{i,t} + \sum \phi_{jT} \ln w_{j,t} + \phi_{TT} t.$$

3. Accounting for GDP Growth

Following Diewert and Morrison's (1986) lead, we define the following *productivity index* in order to capture the GDP effect of the change in technology between time $t - 1$ and time t[11]:

(8) $$R_{t,t-1} \equiv \left[\frac{C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t)}{C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1)} \cdot \frac{C(\mathbf{y}_t, \mathbf{w}_t, t)}{C(\mathbf{y}_t, \mathbf{w}_t, t-1)} \right]^{-\frac{1}{2}}.$$

$R_{t,t-1}$ is defined as the *inverse* of the cost reduction that results from the passage of time, holding output quantities and factor prices constant. The output mix and factor rental prices could equally well be held constant at their period $t - 1$ level or at their period t level. By taking the geometric mean of what may be interpreted as Laspeyres and Paasche productivity indexes, $R_{t,t-1}$ has the Fisher form. A technological improvement that takes place between times $t - 1$ and t, and which acts to reduce total costs, will result in a value of $R_{t,t-1}$ that is greater than one.

Next, consider the GDP effect of a change in the composition of output between times $t - 1$ and t. We can define the following *output quantity effect*, $Y_{i,t,t-1}$ to capture the GDP effect of a change in the quantity of output i between time $t - 1$ and time t:

(9) $$Y_{i,t,t-1} \equiv \left[\frac{C(y_{1,t-1}, \ldots, y_{i,t}, \ldots y_{I,t-1}, \mathbf{w}_{t-1}, t-1) \cdot C(\mathbf{y}_t, \mathbf{w}_t, t)}{C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1) \cdot C(y_{1,t}, \ldots, y_{i,t-1}, \ldots, y_{I,t}, \mathbf{w}_t, t)} \right]^{\frac{1}{2}}, \quad i = 1, \ldots, I.$$

Note that $Y_{i,t,t-1}$ too can be interpreted as the geometric mean of Laspeyres and Paasche indexes.

[11]Through some simple algebraic manipulations, it can be shown that $R_{t,t-1}$ can also be expressed as follows:

$$R_{t,t-1} \equiv \frac{1}{\Gamma_{t,t-1}} \left[\frac{C(\mathbf{y}_t, \mathbf{w}_{t-1}, t-1)}{C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1)} \frac{C(\mathbf{y}_t, \mathbf{w}_t, t)}{C(\mathbf{y}_{t-1}, \mathbf{w}_t, t)} \right]^{\frac{1}{2}} \left[\frac{C(\mathbf{y}_{t-1}, \mathbf{w}_t, t-1)}{C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1)} \frac{C(\mathbf{y}_t, \mathbf{w}_t, t)}{C(\mathbf{y}_t, \mathbf{w}_{t-1}, t)} \right]^{\frac{1}{2}},$$

where $\Gamma_{t,t-1}$ is one plus the rate of change in nominal GDP between periods $t - 1$ and t; see (12) below. I am grateful to the referee for pointing this out to me. See footnotes 13 and 15 for additional details on the interpretation of this expression.

Finally, we consider the GDP contribution of domestic factor rental prices, and we define the following *primary input price effect*:

$$W_{j,t,t-1} \equiv \left[\frac{C(y_{t-1},w_{1,t-1},\ldots,w_{j,t},\ldots,w_{J,t-1},t-1)\cdot C(\mathbf{y}_t,\mathbf{w}_t,t)}{C(\mathbf{y}_{t-1},\mathbf{w}_{t-1},t-1)\cdot C(\mathbf{y}_t,w_{1,t},\ldots,w_{j,t-1},\ldots,w_{J,t},t)}\right]^{\frac{1}{2}}, \quad j=1,\ldots,J. \tag{10}$$

$W_{j,t,t-1}$ thus indicates the growth in GDP that is imputable to changes in the rental price of factor j, holding other factor rental prices, the level of output and the technology constant.

The three factors identified in this section capture the main sources of GDP growth. We should emphasize that expressions (8)–(10) are valid for small and for large changes in prices, quantities, and technology—not merely for infinitesimal variations. In the next two sections, we will turn to the task of actually measuring these effects. Moreover, we will see that, under certain conditions, these three types of effects will give a complete decomposition of observed GDP growth.

4. Index Numbers

The national income function is generally unknown, and thus (8)–(10) are not very useful for obtaining estimates of $R_{t,t-1}$, $Y_{i,t,t-1}$ ($i = 1, \ldots, I$), and $W_{j,t,t-1}$ ($j = 1, \ldots, J$). However, it turns out that *if the national income function has the translog form*, i.e. if $C(\cdot)$ is given by (4), $R_{t,t-1}$ can be calculated from the data alone in the following way[12]:

$$R_{t,t-1} = \frac{Y_{t,t-1}\cdot W_{t,t-1}}{\Gamma_{t,t-1}}, \tag{11}$$

where

$$\Gamma_{t,t-1} \equiv \frac{\sum p_{i,t}y_{i,t}}{\sum p_{i,t-1}y_{i,t-1}} \tag{12}$$

$$Y_{t,t-1} \equiv \exp\left[\sum \frac{1}{2}(s_{i,t}+s_{i,t-1})\ln\frac{y_{i,t}}{y_{i,t-1}}\right] \tag{13}$$

$$W_{t,t-1} \equiv \exp\left[\sum \frac{1}{2}(s_{j,t}+s_{j,t-1})\ln\frac{w_{j,t}}{w_{j,t-1}}\right], \quad i=1,\ldots,I;\ j=1,\ldots,J. \tag{14}$$

[12]See the Appendix for a proof. Note that $R_{t,t-1}$ given by (11) comes close to, but is not identical to, a commonly used measure of total factor productivity growth ($\tilde{R}_{t,t-1}$), where a Törnqvist index of real output is divided by a Törnqvist index of input quantities ($X_{t,t-1}$): $\tilde{R}_{t,t-1} \equiv Y_{t,t-1}/X_{t,t-1}$. Indeed, the ratio $\Gamma_{t,t-1}/W_{t,t-1}$ in (11) can be interpreted as an *implicit* Törnqvist input quantity index, which is numerically different from a straight Törnqvist input quantity index since Törnqvist indexes do not satisfy the Fisher factor reversal test. This implies that the conventional measure is not exact for the translog national income function. The measure proposed by Diewert and Morrison (1986) ($R^{DM} \equiv (\Gamma_{t,t-1}/P_{t,t-1})/X_{t,t-1}$ where $P_{t,t-1}$ is a Törnqvist index of output prices) is not exact for the translog national income function either, although it is exact for the translog GDP function. Numerically these different measures of total factor productivity would normally be very close to each other, though.

$\Gamma_{t,t-1}$ is one plus the rate of increase in nominal GDP between times $t-1$ and t. $Y_{t,t-1}$ is the Törnqvist *output quantity index*: it can be viewed as a Törnqvist chain-type index of real GDP. $W_{t,t-1}$ is the Törnqvist *primary input price index*.

Furthermore, following Diewert and Morrison (1986), one finds that as long as the national income function has the translog form, $Y_{i,t,t-1}$ can be calculated from knowledge of the data alone:

$$Y_{i,t,t-1} = \exp\left[\frac{1}{2}(s_{i,t} + s_{i,t-1})\ln\frac{y_{i,t}}{y_{i,t-1}}\right], \quad i = 1,\ldots,I, \tag{15}$$

whereas $W_{j,t,t-1}$ can be obtained as:

$$W_{j,t,t-1} = \exp\left[\frac{1}{2}(s_{j,t} + s_{j,t-1})\ln\frac{w_{j,t}}{y_{j,t-1}}\right], \quad j = 1,\ldots,J. \tag{16}$$

It can immediately be seen from (12) and (15) that[13]:

$$Y_{t,t-1} = \prod_{i=1}^{I} Y_{i,t,t-1}. \tag{17}$$

The right-hand side of (17) thus provides a multiplicative decomposition of the Törnqvist index of real GDP.[14]

Similarly, one notes that[15]:

$$W_{t,t-1} = \prod_{j=1}^{J} W_{j,t,t-1}. \tag{18}$$

Expressions (11), (17) and (18) taken together imply that, as long as the national income function is translog, the following gives a *complete* and *exact* decomposition of GDP growth[16]:

$$\Gamma_{t,t-1} = \frac{1}{R_{t,t-1}} \cdot \prod_{i=1}^{I} Y_{i,t,t-1} \cdot \prod_{j=1}^{J} W_{j,t,t-1}. \tag{19}$$

[13]Note that if the national income function is translog, the index of real GDP can be expressed as $Y_{t,t-1} = \{C(\mathbf{y}_t, \mathbf{w}_{t-1}, t-1) \cdot C(\mathbf{y}_t, \mathbf{w}_t, t)/[C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1) \cdot C(\mathbf{y}_{t-1}, \mathbf{w}_t, t)]\}^{1/2}$.

[14]If one uses the Laspeyres quantity aggregation, an additive decomposition can easily be obtained. Let $Y^L_{t,t-1} \equiv \Sigma_i y_{i,t}/\Sigma_i y_{i,t-1}$ be one plus the rate of change in the direct Laspeyres index of real GDP. It can then easily be seen that $Y^L_{t,t-1} - 1 = \Sigma_i \omega_{i,t-1}(y_{i,t}/y_{i,t-1} - 1)$, where $\omega_{i,t-1} \equiv y_{i,t-1}/\Sigma_i y_{i,t-1}$. While this decomposition is complete and exact for the Laspeyres aggregation, it is obviously not superlative since the Laspeyres index is not exact for any flexible functional form. Note also that, strictly speaking, $Y^L_{t,t-1}$ is *not* a Laspeyres index, unless $t-1$ happens to be the base period for which all prices have been normalized to one. If one uses the Fisher index of real GDP, one possible decomposition—the one used by the BEA—is given by $Y^F_{t,t-1} - 1 = \Sigma\Omega_{i,t,t-1}(y_{i,t} - y_{i,t-1})$, where $\Omega_{i,t,t-1} \equiv (p_{i,t-1} + p_{i,t}/P^F_{t,t-1})/(\Sigma_i p_{i,t-1} y_{i,t-1} + \Sigma_i p_{i,t} y_{i,t-1}/P^F_{t,t-1})$, and where $Y^F_{t,t-1}$ and $P^F_{t,t-1}$ are the chain-type Fisher indexes of output quantities and prices, respectively; see Moulton and Seskin (1999) and Reinsdorf, Diewert, and Ehemann (2000).

[15]If the national income function is translog, the Törnqvist index of primary input prices can be expressed as $W_{t,t-1} = \{C(\mathbf{y}_{t-1}, \mathbf{w}_t, t-1) \cdot C(\mathbf{y}_t, \mathbf{w}_t, t)/[C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1) \cdot C(\mathbf{y}_t, \mathbf{w}_{t-1}, t)]\}^{1/2}$.

[16]This decomposition is complete in the sense that with $R_{t,t-1}$, $Y_{i,t,t-1}$ and $W_{j,t,t-1}$ defined by (8)–(10) and $\Gamma_{t,t-1}$ by (12), there is no residual. It is exact in the sense that (11), (15) and (16) are exact measures of (8)–(10).

5. Estimates for the United States

We report in Table 1 estimates of the decomposition of U.S. GDP growth based on (19), using annual data for the period 1948–98.[17] We consider five outputs—consumption (C), investment (I), government purchases (G), exports (X) and imports (M)—and two primary factors—labor (L) and capital (K). Geometric averages for ten-year subperiods and for the entire sample period are shown at the bottom of the table. The real GDP index ($Y_{t,t-1}$) and the primary input price index ($W_{t,t-1}$) are also reported.

Focusing first on the figures for the entire period, we find that GDP has increased at an average annual rate of about 7.2 percent. On the output side, it is consumption which has been the driving force, with an average contribution to GDP growth of about 2.2 percent annually. The contribution of investment has reached about 0.6 percent, while government purchases and exports contributed about 0.6 percent and 0.4 percent, respectively. Naturally, imports tend to have an offsetting effect on growth, with an average contribution of about –0.5 percent per year. On balance, the average contribution of foreign trade to U.S. GDP growth has been very slightly negative. The product of these five quantity effects makes up the Törnqvist real GDP index ($Y_{t,t-1}$) which has averaged 3.5 percent over the sample period. On the price side, increases in the return to labor have added nearly 3.6 percent to GDP growth, whereas the contribution of capital has averaged about 1.0 percent. These rather large contributions reflect important increases in factor rental prices made possible in parts by technological progress which has contributed to reduce costs by nearly 1.0 percent annually.

Increases in GDP were largest during the 1970s, when increases in factor payments were strongest. Thus, GDP increased by over 13 percent in 1978, and by nearly 12 percent the following year, when the contribution of the rental price of labor surpassed 6.5 percent. More recently, the price of labor services has had a milder effect on total costs, its contribution often being less than 2 percent.

The contribution of consumption to GDP growth has been positive in all but three years over the sample period. As expected, it tends to be fairly steady, whereas the contribution of investment is much more volatile: it has been as low as –4.2 percent in 1949, and as high as 5.7 percent the following year. The contribution of government has been modest in recent years, while exports have had an increasingly large contribution, although this has been more than offset by imports.

The total factor productivity index $R_{t,t-1}$ was highest the 1950s and the 1960s. This is when technological progress was highest and cost reduction strongest.

[17]The output data are from the Bureau of Economic Analysis: NIPA, Tables 1.1 and 1.2. The shares of capital and labor together with their rental prices are from the Bureau of Labor Statistics multifactor productivity database. These are for the private business sector, and we therefore assume that they do not differ greatly from their values for the economy at large. Note that, in the absence of an adjustment for factor payments to and from abroad, for the consumption of fixed capital, and for indirect taxes and subsidies, the national income identity does not hold. This does not matter, however, since one advantage of the translog functional form is that, next to the output quantities and primary input prices, only the GDP and national income *shares* are required.

TABLE 1

NATIONAL INCOME GROWTH ACCOUNTING: MULTIPLICATIVE DECOMPOSITION

	Y_C	Y_I	Y_G	Y_X	Y_M	Y	W_L	W_K	W	R	Γ
1949	1.0179	0.9581	1.0181	0.9994	1.0013	0.9937	0.9958	0.9793	0.9752	0.9755	0.9933
1950	1.0419	1.0574	1.0000	0.9936	0.9939	1.0880	1.0483	1.0371	1.0873	1.0757	1.0997
1951	1.0094	1.0005	1.0571	1.0094	0.9984	1.0759	1.0673	1.0284	1.0976	1.0241	1.1532
1952	1.0194	0.9838	1.0424	0.9978	0.9964	1.0394	1.0375	0.9884	1.0255	1.0092	1.0563
1953	1.0292	1.0069	1.0161	0.9970	0.9962	1.0458	1.0357	0.9927	1.0281	1.0152	1.0591
1954	1.0128	0.9932	0.9835	1.0020	1.0021	0.9933	1.0149	0.9926	1.0074	0.9972	1.0034
1955	1.0449	1.0340	0.9917	1.0042	0.9953	1.0710	1.0193	1.0397	1.0598	1.0418	1.0895
1956	1.0181	0.9978	1.0001	1.0070	0.9967	1.0197	1.0436	0.9948	1.0382	1.0036	1.0549
1957	1.0151	0.9930	1.0093	1.0042	0.9982	1.0199	1.0415	1.0029	1.0446	1.0109	1.0539
1958	1.0051	0.9873	1.0071	0.9931	0.9980	0.9904	1.0261	0.9932	1.0191	0.9955	1.0139
1959	1.0348	1.0276	1.0124	1.0004	0.9957	1.0722	1.0241	1.0280	1.0527	1.0411	1.0842
1960	1.0170	0.9999	1.0000	1.0084	0.9994	1.0249	1.0239	0.9888	1.0124	0.9981	1.0396
1961	1.0126	0.9990	1.0104	1.0008	1.0003	1.0232	1.0241	1.0019	1.0260	1.0148	1.0345
1962	1.0305	1.0176	1.0132	1.0025	0.9955	1.0604	1.0259	1.0189	1.0452	1.0313	1.0748
1963	1.0254	1.0098	1.0053	1.0034	0.9989	1.0432	1.0240	1.0135	1.0379	1.0264	1.0549
1964	1.0367	1.0122	1.0043	1.0062	0.9978	1.0580	1.0367	1.0103	1.0474	1.0320	1.0739
1965	1.0386	1.0211	1.0067	1.0010	0.9957	1.0640	1.0277	1.0208	1.0491	1.0299	1.0838
1966	1.0347	1.0140	1.0188	1.0032	0.9937	1.0656	1.0478	1.0051	1.0531	1.0240	1.0960
1967	1.0182	0.9925	1.0167	1.0011	0.9967	1.0250	1.0395	0.9901	1.0292	0.9981	1.0569
1968	1.0345	1.0087	1.0074	1.0035	0.9932	1.0477	1.0544	1.0120	1.0670	1.0231	1.0927
1969	1.0225	1.0089	0.9990	1.0026	0.9972	1.0304	1.0441	0.9969	1.0408	0.9920	1.0811
1970	1.0143	0.9897	0.9948	1.0054	0.9978	1.0019	1.0487	0.9850	1.0329	0.9808	1.0551
1971	1.0234	1.0165	0.9958	1.0004	0.9972	1.0334	1.0466	1.0204	1.0679	1.0166	1.0856
1972	1.0371	1.0183	1.0003	1.0042	0.9939	1.0543	1.0451	1.0253	1.0715	1.0279	1.0991
1973	1.0301	1.0192	0.9985	1.0119	0.9972	1.0577	1.0643	1.0205	1.0862	1.0286	1.1170
1974	0.9949	0.9869	1.0038	1.0068	1.0017	0.9941	1.0580	0.9960	1.0537	0.9669	1.0834
1975	1.0134	0.9705	1.0041	0.9994	1.0095	0.9964	1.0656	1.0340	1.1018	1.0078	1.0894
1976	1.0363	1.0280	1.0002	1.0048	0.9860	1.0557	1.0621	1.0275	1.0913	1.0328	1.1155
1977	1.0268	1.0241	1.0021	1.0020	0.9911	1.0464	1.0511	1.0275	1.0800	1.0148	1.1136
1978	1.0275	1.0202	1.0062	1.0080	0.9925	1.0552	1.0599	1.0241	1.0854	1.0134	1.1302
1979	1.0156	1.0059	1.0038	1.0078	0.9985	1.0319	1.0656	1.0101	1.0763	0.9935	1.1179
1980	0.9980	0.9793	1.0039	1.0097	1.0070	0.9976	1.0643	1.0026	1.0671	0.9773	1.0893
1981	1.0084	1.0157	1.0018	1.0011	0.9973	1.0244	1.0558	1.0350	1.0928	0.9994	1.1201
1982	1.0077	0.9746	1.0032	0.9933	1.0012	0.9798	1.0407	1.0026	1.0434	0.9822	1.0408
1983	1.0347	1.0145	1.0069	0.9980	0.9890	1.0433	1.0248	1.0158	1.0410	1.0013	1.0846
1984	1.0342	1.0456	1.0070	1.0063	0.9789	1.0727	1.0349	1.0383	1.0745	1.0361	1.1125
1985	1.0314	0.9983	1.0130	1.0020	0.9937	1.0385	1.0339	1.0020	1.0359	1.0042	1.0713
1986	1.0269	0.9989	1.0112	1.0051	0.9919	1.0342	1.0334	0.9887	1.0216	0.9997	1.0569
1987	1.0215	1.0041	1.0062	1.0080	0.9939	1.0340	1.0213	1.0203	1.0421	1.0117	1.0650
1988	1.0263	1.0043	1.0024	1.0123	0.9960	1.0417	1.0283	1.0266	1.0557	1.0210	1.0772
1989	1.0174	1.0059	1.0055	1.0101	0.9958	1.0351	1.0177	1.0070	1.0249	0.9872	1.0745
1990	1.0121	0.9951	1.0065	1.0079	0.9959	1.0176	1.0318	1.0022	1.0340	0.9953	1.0572
1991	0.9988	0.9874	1.0024	1.0062	1.0005	0.9953	1.0204	0.9960	1.0163	0.9806	1.0315
1992	1.0189	1.0111	1.0010	1.0060	0.9933	1.0305	1.0295	1.0063	1.0360	1.0113	1.0556
1993	1.0223	1.0118	0.9984	1.0033	0.9908	1.0265	1.0130	1.0086	1.0218	0.9978	1.0512
1994	1.0251	1.0187	1.0002	1.0087	0.9875	1.0403	1.0075	1.0224	1.0301	1.0090	1.0620
1995	1.0200	1.0046	1.0008	1.0105	0.9907	1.0267	1.0162	1.0078	1.0242	1.0023	1.0491
1996	1.0213	1.0135	1.0020	1.0088	0.9899	1.0357	1.0182	1.0130	1.0315	1.0119	1.0558
1997	1.0236	1.0188	1.0042	1.0133	0.9841	1.0443	1.0212	1.0085	1.0298	1.0102	1.0646
1998	1.0310	1.0204	1.0037	1.0025	0.9859	1.0436	1.0350	0.9940	1.0288	1.0160	1.0567
1949–58	1.0213	1.0009	1.0123	1.0008	0.9977	1.0332	1.0328	1.0047	1.0377	1.0145	1.0567
1959–68	1.0283	1.0102	1.0095	1.0030	0.9967	1.0483	1.0327	1.0089	1.0419	1.0218	1.0689
1969–78	1.0226	1.0081	1.0005	1.0045	0.9964	1.0322	1.0545	1.0156	1.0710	1.0079	1.0968
1979–88	1.0204	1.0040	1.0059	1.0043	0.9947	1.0295	1.0402	1.0141	1.0548	1.0025	1.0833
1989–98	1.0190	1.0087	1.0025	1.0077	0.9914	1.0295	1.0210	1.0065	1.0277	1.0021	1.0558
1949–98	1.0223	1.0064	1.0061	1.0041	0.9954	1.0345	1.0362	1.0100	1.0465	1.0097	1.0722

Explanations:
$Y_{i,t,t-1}$: output quantity effect i ($i = C, I, G, X, M$)
$Y_{t,t-1}$: Törnqvist index of real GDP
$W_{j,t,t-1}$: primary input price effect j ($j = L, K$)
$W_{t,t-1}$: Törnqvist index of primary input prices
$R_{t,t-1}$: total factor productivity index
$\Gamma_{t,t-1}$: nominal GDP growth.
The t and $t - 1$ subscripts are omitted for the sake of conciseness.

Technological progress fell dramatically during the 1970s, but it has increased again in recent years.

6. Concluding Comments

The decomposition of GDP that is proposed in this paper is complete and exact as long as the true national income function does indeed have the translog form. Since the translog functional form is flexible, the index number approach used here is superlative in the sense of Diewert (1976).

Compared to the method used by the BEA, our approach seems to have several advantages. First, the Törnqvist aggregation is exact for a functional form—the translog—that is capable of providing a second-order approximation to an arbitrary national income function without requiring weak separability between outputs and primary inputs. As discussed earlier, the Fisher index used by the BEA is also a superlative index,[18] but the functional form for which it is exact, the square-rooted quadratic, does not seem to be well suited to approximate a national income function, except in the restrictive case where outputs are globally separable.[19] Second, the Törnqvist decomposition is much simpler than the one used by the BEA, and each one of its terms has an economic interpretation as indicated by (8)–(10), which is not true for the Fisher decomposition. Third, our decomposition goes beyond the various output components and real GDP: by accounting for the effects of domestic factor rental prices and technological change, it can provide a complete decomposition of nominal GDP growth.[20] Finally, it provides a multiplicative decomposition, rather than an additive one; this might be appealing to economists who are used to compound growth rates, rather than adding them up.[21]

While we have used an index number approach in this paper, one could also opt for an econometric approach. That is, if econometric estimates of the parameters of (4) were available, the terms given by (8)–(10) could be computed

[18]One common feature of the Törnqvist and the Fisher indexes is that neither one is consistent in aggregation, although this is generally viewed as being of little numerical significance.

[19]In that case, the national income function could be written as $C(\mathbf{y}_t, \mathbf{w}_t, t) = (\mathbf{y}_t' A \mathbf{y}_t)^{1/2} c(\mathbf{w}_t, t)$, where A is positive semi-definite matrix; see footnote 9.

[20]That is, the BEA decomposition only deals with the term $(\mathbf{y}_t' A \mathbf{y}_t)^{1/2}$ if one refers to footnote 19.

[21]The third decomposition of the Fisher index proposed by Reinsdorf, Diewert, and Ehemann (2000) is based on the additive decomposition of a geometric mean. Interestingly enough, this suggests that the Fisher index can also be expressed as a geometric mean, in which case a multiplicative decomposition is straightforward. One can thus show that the Fisher index of real GDP ($Y^F_{t,t-1}$) can be written as:

$$Y^F_{t,t-1} = \prod_i \left(\frac{y_{i,t}}{y_{i,t-1}} \right)^{\sigma_{i,t}},$$

where

$$\sigma_{i,t} = \frac{1}{2} \left[\frac{s_{i,t-1} m(y_{i,t}/y_{i,t-1}, Y^L_{t,t-1})}{\Sigma_j s_{j,t-1} m(y_{j,t}/y_{j,t-1}, Y^L_{t,t-1})} + \frac{s_{i,t} m(y_{i,t-1}/y_{i,t}, 1/Y^P_{t,t-1})}{\Sigma_j s_{j,t} m(y_{j,t-1}/y_{j,t}, 1/Y^P_{t,t-1})} \right],$$

with $m(a, b) \equiv (a - b)/(\ln a - \ln b)$ for $a, b > 0$, and where $Y^L_{t,t-1}$ and $Y^F_{t,t-1}$ are, respectively, the Laspeyres and Paasche indexes of real GDP. We have verified that this decomposition gives results that are numerically very close to ours.

directly.[22] The advantage of the index number approach, however, is that it is much simpler, and by not depending on the choice of a stochastic specification and of an estimation technique, it seems less subjective and, hence, less controversial.

Appendix

The purpose of this appendix is to provide a proof of (11). It is useful to begin by expressing the change in nominal GDP in terms of national income function (4):

$$\begin{aligned}
\ln \Gamma_{t,t-1} &= \ln C(\mathbf{y}_t, \mathbf{w}_t, t) - \ln C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1) \\
&= \sum_i \alpha_i (\ln y_{i,t} - \ln y_{i,t-1}) + \sum_i \beta_j (\ln w_{j,t} - \ln w_{j,t-1}) \\
&\quad + \frac{1}{2} \sum_i \sum_h \gamma_{ih} (\ln y_{i,t} \ln y_{h,t} - \ln y_{i,t-1} \ln y_{h,t-1}) \\
&\quad + \frac{1}{2} \sum_j \sum_k \phi_{jk} (\ln w_{j,t} \ln w_{k,t} - \ln w_{j,t-1} \ln w_{k,t-1}) \\
&\quad + \sum_i \sum_j \delta_{ij} (\ln y_{i,t} \ln w_{j,t} - \ln y_{i,t-1} \ln w_{j,t-1}) \\
&\quad + \sum_i \delta_{iT} [\ln y_{i,t} t - \ln y_{i,t-1} (t-1)] + \sum_j \phi_{jT} [\ln w_{j,t} t - \ln w_{j,t-1} (t-1)] \\
&\quad + \beta_T + \frac{1}{2} \phi_{TT} \left[t^2 - (t-1)^2 \right] \\
&= \sum_i \alpha_i (\ln y_{i,t} - \ln y_{i,t-1}) + \sum_i \beta_j (\ln w_{j,t} - \ln w_{j,t-1}) \\
&\quad + \frac{1}{4} \sum_i \sum_h \gamma_{ih} [(\ln y_{i,t} + \ln y_{i,t-1})(\ln y_{h,t} - \ln y_{h,t-1}) \\
&\quad + (\ln y_{h,t} + \ln y_{h,t-1})(\ln y_{i,t} - \ln y_{i,t-1})] \\
&\quad + \frac{1}{4} \sum_j \sum_k \phi_{jk} [(\ln w_{j,t} + \ln w_{j,t-1})(\ln w_{k,t} - \ln w_{k,t-1}) \\
&\quad + (\ln w_{k,t} + \ln w_{k,t-1})(\ln w_{j,t} - \ln w_{j,t-1})] \\
&\quad + \frac{1}{2} \sum_i \sum_j \delta_{ij} [(\ln y_{i,t} + \ln y_{i,t-1})(\ln w_{j,t} - \ln w_{j,t-1}) \\
&\quad + (\ln w_{j,t} + \ln w_{j,t-1})(\ln y_{i,t} - \ln y_{i,t-1})] \\
&\quad + \frac{1}{2} \sum_i \delta_{iT} [(\ln y_{i,t} + \ln y_{i,t-1}) + (2t-1)(\ln y_{i,t} - \ln y_{i,t-1})] \\
&\quad + \frac{1}{2} \sum_j \phi_{jT} [(\ln w_{j,t} + \ln w_{j,t-1}) + (2t-1)(\ln w_{j,t} - \ln w_{j,t-1})] \\
&\quad + \beta_T + \frac{1}{2} \phi_{TT} (2t-1)
\end{aligned}$$

[22]See Kohli (1990) for additional details in the context of the GDP function.

$$
\begin{aligned}
&= \frac{1}{2}\sum_i (s_{i,t} + s_{i,t-1})(\ln y_{i,t} - \ln y_{i,t-1}) \\
&\quad + \frac{1}{2}\sum_j (s_{j,t} + s_{j,t-1})(\ln w_{j,t} - \ln w_{j,t-1}) + \frac{1}{2}(s_{T,t} + s_{T,t-1}) \\
&= \ln Y_{t,t-1} + \ln W_{t,t-1} + \frac{1}{2}(s_{T,t} + s_{T,t-1}),
\end{aligned}
$$

where we have made use of (13) and (14).

Next, using (8) as a starting point, and substituting (4), we find that:

$$
\begin{aligned}
\ln R_{t,t-1} &= -\frac{1}{2}[\ln C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t) - \ln C(\mathbf{y}_{t-1}, \mathbf{w}_{t-1}, t-1) \\
&\quad + \ln C(\mathbf{y}_t, \mathbf{w}_t, t) - \ln C(\mathbf{y}_t, \mathbf{w}_t, t-1)] \\
&= -\frac{1}{2}\Bigg\{\sum_i \delta_{iT} \ln y_{i,t-1} + \sum_j \phi_{jT} \ln w_{j,t-1} + \beta_T + \frac{1}{2}\phi_{TT}\left[t^2 - (t-1)^2\right] \\
&\quad + \sum_i \delta_{iT} \ln y_{i,t} + \sum_j \phi_{jT} \ln w_{j,t} + \beta_T + \frac{1}{2}\phi_{TT}\left[t^2 - (t-1)^2\right]\Bigg\} \\
&= -\frac{1}{2}\Bigg[\beta_T + \sum_i \delta_{iT} \ln y_{i,t} + \sum_j \phi_{jT} \ln w_{j,t} + \phi_{TT} t \\
&\quad + \beta_T + \sum_i \delta_{iT} \ln y_{i,t-1} + \sum_j \phi_{jT} \ln w_{j,t-1} + \phi_{TT}(t-1)\Bigg] \\
&= -\frac{1}{2}(s_{T,t} + s_{T,t-1}) \\
&= \ln Y_{t,t-1} + \ln W_{t,t-1} - \ln \Gamma_{t,t-1},
\end{aligned}
$$

which is equivalent to (11). Note that there is no need to assume global separability between inputs and outputs (which would require $\delta_{ij} = \delta_{iT} = 0, \forall i, j$) to obtain this result.

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