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AN ALTERNATIVE AXIOMATIZATION OF SEN'S POVERTY MEASURE

BY PRASANTA K. PATTANAIK
University of California (Riverside)

AND

MANIMAY SENGUPTA
University of Canterbury

We provide an alternative axiomatization of Sen's (1976) poverty measure. We derive the measure from the general definition of a poverty measure by using a version of Sen's rank order axiom, and a substantially weaker form of his normalization axiom. These two axioms, together with a continuity axiom, lead uniquely to the measure suggested by Sen (1976).

1. INTRODUCTION

In his seminal work on poverty, Sen (1976) suggested a measure of poverty. The central point of departure of this work was to take note of the inequality in the distribution of income among the poor. Sen's work has led to a large number of contributions in the literature on the measurement of poverty [surveys of this literature may be found in Foster (1984), Hagenaars (1987), Kanbur (1984), Seidl (1988) and Sen (1979, 1981)]. Also, Sen's poverty measure has been widely used empirically.

In this paper, we provide an axiomatization of Sen's poverty measure. Sen (1976) provided an axiomatic derivation of his measure. However, there are some limitations in regard to the axiomatization suggested by Sen (1976). Firstly, Sen (1976) starts with a definition of poverty, in which poverty is taken as a normalized weighted sum of the income shortfalls of the poor from the poverty line. This definition, together with an axiom specifying the weights by the rank order of the poor in terms of their incomes, and a normalization axiom, leads to the unique functional form for the poverty measure. Defined on the incomes of the poor, the measure has four main properties: monotonicity in the incomes of the poor, strict S-convexity in these incomes which lends the measure its inequality-sensitivity, and two technical but convenient properties of continuity in the incomes of the poor and a normalization in the unit interval. Aside from the fact that the use of the *a priori* definition of poverty leaves the linear form of the measure unjustified, none of these properties of the measure are implied by the axioms themselves in the absence of the definition of poverty used. The use of the definition of poverty thus obscures the role of Sen's central axioms in the derivation of the poverty measure, for it does not allow it to be made clear as to what extent the axioms are responsible in inducing these properties for the measure.

Secondly, in deriving his poverty measure, Sen (1976) uses a normalization axiom which specifies that, in a situation where the poor all have equal incomes, the poverty value is given by the product of the "head-count ratio" and the "income-gap ratio." Several authors, including Sen (1976), have noted that the multiplicative form chosen for the axiom was arbitrary.[1,2] However, apart from the particular form the normalization axiom takes, there is a prior problem in the intuitive motivation for the axiom. Sen (1976) justifies the axiom by claiming that "in the special case in which all the poor have exactly the *same* income level, $y^* < z$, it can be argued that H and I together should give us adequate information on the level of poverty, since in this special case the two together can tell us all about the proportion of people who are below the poverty line and the extent of the income shortfall of each" [Sen (1976), p. 223]. However, as we have argued elsewhere [Pattanaik and Sengupta (1993)], defining poverty in terms of H and I is neither necessary nor sufficient for deriving an appropriate measure of poverty for the special case of equal incomes of the poor.

The purpose of this paper is to provide an axiomatization of Sen's measure that dispenses with both of the two arbitrary features of the *a priori* definition of poverty and the normalization axiom in the derivation of Sen's measure. The motivation for our alternative axiomatization arises from the wide empirical use of the measure. We derive the measure from the general definition of a poverty measure defined on the incomes of the population, placing restrictions on the measure in terms of our axioms. We show that a reformulation of Sen's rank order axiom, together with an intuitively transparent normalization axiom and an axiom of continuity of the measure in the incomes of the poor can be used to characterize the measure. The normalization axiom we use is an axiom for restricting the measure in the unit interval. Moreover, given the continuity of the measure, the rank order axiom we specify is seen to imply strict S-convexity of the measure, while these three axioms together entail monotonicity of the measure in the incomes of the poor. The axioms we use thus lead directly to the main properties of the Sen measure, and also help to clarify the role of Sen's (1976) main axioms in the derivation of his poverty measure. Moreover, our derivation of the measure from the general definition of a poverty measure extends the empirical applications of the measure.

2. The Notation and the Axioms

Let S be the set of individuals in the economy, with n members. The individuals in the economy are indexed by $i = 1, \ldots, n$. We denote by z the *poverty line* for the economy. $y = (y_1, \ldots, y_n)$ stands for a vector of incomes of the individuals. An individual $i \in S$ is poor if and only if $y_i \leq z$. Given z and y, $Q =$

[1]The "head-count ratio" (H) is the proportion of the poor in the population and the "income-gap ratio" (I) is the percentage of the aggregate income shortfall of the poor from the poverty line. Using the notation introduced below (p. 3) we have:

$$\mathrm{H} = \frac{q}{n} \quad \text{and} \quad \mathrm{I} = \sum_{i \in Q} \frac{(z - y_i)}{qz}.$$

[2]The normalization axiom has been discussed and evaluated by Anand (1977), Basu (1985) and Thon (1979).

$\{i \in S \mid y_i \le z\}$ denotes the set of the poor, and y_Q the vector of incomes of the poor, the restriction of y to Q. $q, 0 \le q \le n$, will denote the number of the poor in the economy.

A poverty measure P is a real-valued function of the integer-valued variable n, and the continuous-valued non-negative variables z and y_i, $i = 1, \ldots, n$. In what follows, without loss of generality, we shall treat n and z as parameters, whose values will remain fixed throughout our discussion. Thus, we have: $P: R^n_+ \rightarrow R$. We write $P = P(y)$.

We are concerned with a class of measures Sen (1981) has called "focused" poverty measures, whose values are independent of the incomes of the nonpoor. Formally, we have:

Focus Axiom. Let $y, y' \in R^n_+$ be any two income vectors and let Q and Q' be the corresponding sets of the poor. If $y_Q = y'_{Q'}$, then $P(y) = P(y')$.

Throughout, we shall assume that the poverty measures under consideration satisfy the Focus Axiom.

We now introduce our axioms. Our first axiom corresponds to Sen's Axiom R ('Ordinal Rank Weights'), in that it specifies a relationship between the poverty measure and the marginal weights on the incomes of the poor in terms of their income ranks. In order to introduce the axiom, we need some definitions.

Let $y \in R^n_+$. For $i \in Q$, we define $r_i(y)$, called the *rank of i in y*, to be r, if y_i is the r-th highest income in the set $\{y_i \le z \mid i \in Q\}$, ties broken arbitrarily. The *marginal poverty weight for i in y*, denoted $p_i(y)$, is defined as follows: for $h \neq 0$, we have:

$$p_i(y) = \frac{(\Delta P)_i}{\Delta y_i},$$

where $(\Delta P)_i = P(y_1, \ldots, y_i + h, \ldots, y_n) - P(y_1, \ldots, y_i, \ldots, y_n)$, and $\Delta y_i = (y_i + h) - y_i$. The marginal poverty weights of the poor specify the response of the poverty measure with respect to a change in their incomes.

Axiom 1. Let $y \in R^n_+$. Given other things, for all $i \in Q$, $p_i(y)$, the marginal poverty weight for i in y, is proportional to $r_i(y)$, the rank of i in y. The proportionality factor is identical for all $j, j \in Q$, and independent of all $y_j, j \in Q$.

Given the *ceteris paribus* assumption, Axiom 1 specifies that, as long as a change in the income of a poor individual i leaves the rank order, and, consequently, the number of the poor unchanged, the marginal weight on y_i is given by the rank of i in y, except for a proportionality factor that may depend on the parameters n, z and the given value of q, but not on the incomes of the poor. Given n, z and q, the proportionality factor is therefore constant in the relevant range of income variation. Thus, Axiom 1 essentially demands that the weights on the incomes of the poor are constant, in the range of variation of the incomes that leaves the ranks of the poor unaltered as a result of any income change.

Let us note the differences between Axiom 1 and Sen's Axiom R. First, given Sen's definition of poverty, and given his Axiom M ("Monotonic Welfare") that specifies greater welfare with higher incomes, Axiom R requires that the marginal poverty weights on the income gaps $(z - y_i)$ of the poor be given by the respective income ranks. Axiom 1 on the other hand specifies these weights on the incomes

y_i of the poor to be proportional to the corresponding ranks. Secondly, Axiom R does not, by itself, require that the weights be proportional to the ranks in the manner specified by Axiom 1; this requirement enters through the definition of poverty Sen uses. Formally, given Sen's (1976) definition of poverty, Axiom 1 is a weaker assumption, implied by Axiom R, though, in general, neither axiom implies the other. Intuitively, the two axioms are similar. Central to both is the idea that "a greater value should be attached to an increase in income (or reduction of shortfall) of a poorer person than that of a relatively richer person" [Sen (1976, p. 221)]. Both Axiom 1 and Axiom R capture this idea by requiring that the weights on the income of a poor individual be given by the person's poverty rank. However, the rank order property in Axiom R was derived from a background—the notion of relative deprivation being central to it—which was shown by Sen (1976) to be the motivation behind the use of the property in specifying the form of a general class of weights on the income shortfalls of the poor. Axiom 1, on the other hand, invokes this property directly to specify the weights on the incomes of the poor. In this sense, Axiom 1 may be considered to be more demanding from an intuitive point of view.

Our next axiom is a normalization axiom.

Axiom 2. For all $y \in R^n_+$: (2.1) if for all $i \in Q$, $y_i = z$, $P(y) = 0$; and (2.2) if for all $i \in Q$, $y_i = 0$, $P(y) = q/n$.

Axiom 2 requires that when all the poor individuals have an income equal to the poverty line income z, the poverty measure equals zero; and when they all have zero income, the poverty measure equals the proportion of the poor in the population. Note that the second part of Axiom 2 implies that when everyone in the economy is poor and has zero income, the poverty measure equals 1. If the measure is decreasing in the incomes of the poor—this monotonicity property will be seen to follow from our axioms—Axiom 2 essentially amounts to a zero-one normalization of the poverty measure. Also, the specification of the normalized values for the poverty measure in both parts of the axiom is intuitively clear. In the first part, when poverty is minimal with respect to incomes, the poverty value is set equal to zero; in the second part, when poverty is extreme, it is reasonable to specify that in such situations the only measure required to capture the extent of the poverty is the simple head count. Given the monotonicity of the poverty measure in the incomes of the poor, the normalization in the second part of the axiom also provides a benchmark for an appraisal of the extent of poverty: for any non-empty set of the poor, the closer the measure is to the head count, the more extreme is the poverty in the economy.

Axiom 2 may be contrasted with Sen's normalization axiom (Axiom N):

Axiom N. Let $y \in R^n_+$. If the poor have the same income, say t_y, $0 \leq t_y \leq z$, then

$$P(y) = \frac{q}{n}\left[\frac{z - t_y}{z}\right].$$

Thus, Axiom N specifies that, when the poor all have the same income, the poverty value be given by the product of the head-count ratio and the income-gap ratio (restricted to the equal incomes of the poor). Sen's normalization axiom clearly implies ours. Essentially, whereas Axiom N gives the poverty value in terms of a specific functional form of the measure whenever the poor have equal incomes, Axiom 2 specifies the value of the poverty measure only at the two endpoints of equal incomes, namely, when the poor all have the minimal income zero, and when they all have the income equal to the poverty line income z.

Our last axiom is one of continuity of the poverty measure in the incomes of the poor.

For $0 \leq q \leq n$, let $D(q) = \{y \in R^n_+ \mid \text{card } Q = q\}$ be the set of all income vectors y such that the number of the poor is exactly q.

Axiom 3. For each $q, 0 \leq q \leq n$, the poverty measure is continuous in $D(q)$.

The continuity property for a poverty measure specified by Axiom 3 ensures that the value of the measure does not change abruptly with small changes in the incomes of the poor. Defined on the incomes of the poor, continuity of a poverty measure is clearly an unexceptionable requirement. In fact, for a poverty measure satisfying the Focus Axiom, continuity in these incomes would seem to be almost a necessary condition for the measure to be acceptable; it seems difficult to envisage a situation where acting discontinuously on these incomes would be a desirable requirement for such a measure. The rank order axiom in Sen (1976), together with his definition of poverty, incorporates the continuity of the poverty measure in these incomes.

3. Characterization of the Poverty Measure

We now characterize Sen's (1976) measure in terms of our axioms. We show that Axiom 1, the modified version of Sen's rank order axiom, leads to a system of partial differential equations whose solution provides the linear structure of the measure in a dense subset of the domain of the measure. Axiom 2 essentially gives "initial conditions" that provide a unique solution to the system, given by the measure. The measure is then extended to the entire domain by the continuity of the measure as specified in Axiom 3. The algebraic details for the solution to the system of partial differential equations are considerably simplified by using a linearity argument, and this is what we do in the proof of the theorem that follows.

In proving the theorem, we shall use the following lemma.

Lemma 1. Let X and Y be metric spaces and let T be a subset of X that is dense in X. Let f and g be continuous functions from X into Y. If for all $x \in T$, $f(x) = g(x)$, then $f(x) = g(x)$ for all $x \in X$.

The proof of the lemma is straightforward and is omitted.

Theorem 1. Let $P = P(y)$ be a poverty measure satisfying the Focus Axiom. P satisfies Axioms 1, 2 and 3 if and only if it is the Sen (1976) measure:

(1) $$P(y) = \frac{2}{(q+1)nz} \sum_{i \in Q} (z - y_i) r_i(y).$$

Proof. That the measure P specified by (1) satisfies Axioms 1, 2 and 3 is clear. We shall show that if a poverty measure $P^*=P^*(y)$ satisfying the Focus Axiom satisfies Axioms 1–3, then $P^*=P$.

Let q, $0\leq 1\leq n$, be fixed. If $q=0$, the proof is trivial, for in this case the assumption that for all $i\in Q$, $y_i=z$ is trivially satisfied and we have $P^*(y)=P(y)=0$, as required by Axiom 2.1. We may assume, then, for the rest of the proof, $q>0$. For the purpose of the proof, we first consider P^* defined on a subset $\bar{D}(q)\subset D(q)$ such that $\bar{D}(q)=\{y\in R^n_+ \mid y_i, y_j>0 \;\&\; y_i\neq y_j \text{ for all } i,j\in Q,\ i\neq j\}$. We shall first show that $P^*=P$ in the domain $\bar{D}(q)$, and then show $P^*=P$ in $D(q)$. Since q is arbitrary, this will prove the theorem.

Let $y\in\bar{D}(q)$. Since for all $i,j\in Q$, $i\neq j$, we have $y_i\neq y_j$, it is clear that there exists $\varepsilon>0$ such that for each $i\in Q$, a change in income from y_i to y_i+h, $|h|\leq\varepsilon$, will leave the ranks r_j unchanged for all $j\in Q$. By Axiom 1 we then have:

$$p_i(y)=\frac{(\Delta P^*)_i}{\Delta y_i}=\theta(n,z,q)r_i(y),\qquad i\in Q, \tag{2}$$

where $(\Delta P^*)_i=P^*(y_1,\ldots,y_i+h,\ldots,y_n)-P^*(y_1,\ldots,y_i,\ldots,y_n)$, and $\Delta y_i=(y_i+h)-y_i=h$, and where $\theta(n,z,q)$ is a factor of proportionality. Letting $h\to 0$ in (2), and noting that the right-hand side of (2) is constant in the limiting process, it is clear that P^* is differentiable with respect to y_i, $i\in Q$. Hence, restricting attention to an appropriate neighbourhood of y, we have:

$$\frac{\partial P^*}{\partial y_i}=\theta(n,z,q)r_i(y),\qquad i\in Q. \tag{3}$$

Equation (3) implies that P^* is linear in y_i, $i\in Q$, and has the form:[3]

$$P^*=\sum_{i\in Q}\theta(n,z,q)r_iy_i+\beta(n,z,q). \tag{4}$$

From Axiom 2.1, $P^*(y)=0$ if $y_i=z$ for all $i\in Q$. Using this, we get from (4):

$$\begin{aligned}\beta(n,z,q)&=-\theta(n,z,q)\sum_{i\in Q}r_iz\\&=-\theta(n,z,q)\frac{q(q+1)}{2}z.\end{aligned} \tag{5}$$

Now taking $y_i=0$ for all $i\in Q$, and using Axiom 2.2, we get from (4) and (5):

$$\theta(n,z,q)=-\frac{2}{(q+1)nz}. \tag{6}$$

From (4)–(6) it follows that $P^*=P$ in the domain $\bar{D}(q)$.

[3]Note that the integrability conditions for the existence of the solution of the system in (3) hold, the general solution to which is provided by equation (4) that follows. The local uniqueness of the solution (given the appropriate boundary conditions) follows from Cauchy–Kowalewski Theorem, given that P^* in (3) is clearly linear, and hence real analytic. This local solution is also a global solution in the appropriate domain, since the point at which (3) holds is arbitrary. See, for example, Courant and Hilbert (1962) or Garabedian (1964).

We now show that $P^* = P$ in the domain $D'(q) \subset D(q)$ such that $D'(q) = \{y \in R^n_+ \mid y_i = y_j \text{ for some } i, j \in Q.\ i \neq j\} \cup \{y \in R^n_+ \mid y_i = 0 \text{ for some } i \in Q\}$. Notice that the domain $D(q)$, $D(q) = \bar{D}(q) \cup D'(q)$, of P^* (and of P) is the closure of $\bar{D}(q)$. Moreover, P is a continuous function in $D(q)$, as is P^* by Axiom 3. Since P^* and P are both continuous in $D(q)$, it follows from Lemma 1 that $P^* = P$ in $D'(q)$. This completes the proof. □

Theorem 1 gives a characterization of Sen's (1976) poverty measure in terms of Axioms 1–3. It is clear from the axiomatization that the main properties of the measure, namely, those of monotonicity, strict S-convexity, continuity, and the normalization of the measure in the unit interval, are implied by these axioms. More specifically, it can be shown that, given continuity, which has been directly used in our characterization in Axiom 3, Axiom 1 implies the property of strict S-convexity for the measure. Similarly, in the presence of Axiom 1 and Axiom 3, Axiom 2 implies the monotonicity property of the measure. Given monotonicity, Axiom 2 also implies that the measure is normalized in [0, 1]. Thus, the central properties of the measure are seen to follow directly from the axioms used to characterize the measure.

4. Concluding Remarks

In this paper, we have provided an alternative characterization of the poverty measure suggested by Sen (1976). The characterization clarifies two important issues in Sen's (1976) axiomatic framework for the derivation of his measure; first, the use of the definition of poverty in specifying the functional form of the measure; and second, the normalization axiom which is used to uniquely specify the measure. It is seen that Axiom 1, which is essentially Sen's rank order axiom, can, by itself, specify the functional form of the measure. Furthermore, given this functional form, a clearly defensible normalization of the measure and a continuity requirement lead uniquely to the measure suggested by Sen (1976).

Our axiomatization here demonstrates the central role that the rank order axiom plays in Sen's system. The main focus of Sen's arguments rests on this axiom—it is this axiom that introduces the notion of relative deprivation in Sen's measure, and makes the measure inequality-sensitive. On the other hand, given the functional form specified by the definition of poverty in Sen (1976), the role of the rank order axiom is minimal in the derivation of the measure: the axiom is used only in specifying the weights on the income shortfalls of the poor; beyond this, it plays no role whatever in the overall derivation of the measure, even though it is in fact the focal axiom in the justification of the measure. The alternative axiomatization given here shows that the rank order axiom is not only at the heart of the conceptual framework of Sen's measure, but it is also powerful enough to precipitate the formal structure of the measure.

The axiomatization given here also shows that the specification of the normalization axiom used in Sen (1976) can be replaced by an intuitively transparent one. The normalization axiom (Axiom 2) we use here has a dual role: on one hand it normalizes the measure in the unit interval; on the other hand it uses the head-count ratio as an intuitively appealing way of measuring poverty in situations of extreme poverty. This last part of the axiom has the additional use in that it allows the head-count ratio to be used as a benchmark for gauging how far the current poverty is from that of the level of maximal poverty of the poor. This is clearly a very convenient property of Sen's measure, which Axiom 2 highlights.

Thus, the alternative characterization of Sen's (1976) measure in this paper clarifies the basic structure of the measure, and highlights its central features. Both for an intuitive understanding of the measure, as well as for the interpretation of the measure in empirical applications, this clarification, we believe, is of interest.

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