THE ESTIMATION OF PRICE EFFECTS IN A SOCIAL ACCOUNTING MATRIX

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The estimation of true basic prices in a Social Accounting Matrix (SAM) has long been recognized as necessary in order to achieve uniform valuation of inputs for meaningful manipulation of the input-output table contained in a SAM, in order to assess the real effects of changes in demand, etc. In practice, approximate basic prices only have normally been calculated in order to avoid matrix inversion among other things. It is equally the case that true basic prices are required if one wishes to assess the price-raising effects of commodity taxes. It is shown through an example that approximate basic prices, as conventionally calculated, are inadequate and potentially misleading for this as indeed they are for achieving uniformity of valuation. There are also problems with the present procedure for calculating true basic prices.

An alternative method of calculating true basic prices is given, which has various advantages over the existing method, and a new approximate method is also derived which appears to represent a definite improvement on the present method. For the main purpose of the paper, however, the prices of concern are those charged by producers to which the methodology equally applies.

1. INTRODUCTION

The stimulus for this paper originally arose from concern over the treatment of commodity and other indirect taxes in Table 2.1 of the United Nations System of National Accounts (S.N.A.), which illustrates the complete system in the form of a social accounting matrix. It would seem from published S.A.M.'s that the SNA procedure has normally been followed and that their compilers might be unaware of the possible order of magnitude of errors thereby introduced. For the time being we will confine our discussion to the case where commodity taxes are introduced since the generalization from this to all forms of price change in production activities is immediate.

Without loss of generality, but for the sake of simplicity, assume that each industry produces only its characteristic product.

The commodity–industry matrix is then the commodity–commodity and the industry–industry matrix.

The matrix can be written as follows:

		Industry			Final Demand
		1(A)	2(B)	3(C)	
Commodity	A(1)	U_{11}	U_{12}	U_{13}	FD_1
	B(2)	U_{21}	U_{22}	U_{23}	FD_2
	C(3)	U_{31}	U_{32}	U_{33}	FD_3
	Imports	M_1	M_2	M_3	FD_m
	Factor Income	W_1	W_2	W_3	
	Total	$q_1 = g_1$	$q_2 = g_2$	$q_3 = g_3$	

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The interpretation of this table is that industry $1(A)$ uses U_{11} of its own product, commodity $A(1)$, industry 2(B) uses U_{12} of commodity $A(1)$, industry 3(C) uses U_{13} whilst FD_1 of the production of commodity $A(1)$ is taken by final demand. Imports, M_i are those used as intermediate inputs by the respective industries. Because of our assumption that each industry produces only its characteristic product, the domestic output of each commodity, q_i is equal to the domestic output of the corresponding industry, g_i. Prices are producers' prices and also basic prices since, for the time being we have assumed that there are no commodity taxes. Again without loss of generality, we will assume that all prices are equal to one, so that q_i refers both to quantities and values.

Dividing the elements of each industry column in the above by its appropriate q_i we obtain the following coefficients, expressed in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

$$[m_1 \quad m_2 \quad m_3] = m'$$

$$[w_1 \quad w_2 \quad w_3] = w'$$

$$[1 \quad 1 \quad 1] = p'$$

A simple numerical example may be of assistance. In this we have assumed that exports are zero, so that final demand is allocated between households, government and capital formation:

TABLE 1

A NUMERICAL EXAMPLE

		Industry			Final Demand		
		1(*A*)	2(*B*)	3(*C*)	Households (C_h)	Government (C_g)	Capital Formation (K)
	A(1)	10	50	30	10	—	—
Commodity	*B*(2)	20	10	60	90	10	10
	C(3)	30	20	30	100	60	60
Imports	(*M*)	20	20	40	30	—	30
Factor Income	(F.I.)	20	100	140	—	20	—
Total		100	200	300	230	90	100

Note that from our national accounting identities, Gross Domestic Product (Y) is given by

$$Y = C_h + C_g + K + (Z - M) = \Sigma(\text{F.I.})$$
$$= 230 + 90 + 100 + (0 - 140) = 280$$

where 'Z' stands for exports.

The matrix and vector coefficients for this example are

$$A=\begin{bmatrix}0.10 & 0.25 & 0.10\\0.20 & 0.05 & 0.20\\0.30 & 0.10 & 0.10\end{bmatrix}\qquad w=\begin{bmatrix}0.20\\0.50\\0.46\end{bmatrix}\qquad m=\begin{bmatrix}0.20\\0.10\\0.13\end{bmatrix}$$

As already mentioned, the true basic prices, b, are equal to the producers' prices, p, because there are no commodity taxes in the example. Thus:

$$b=p=A'p+w+m=[I-A']^{-1}(w+m) \tag{1}$$

Suppose excise duties and other commodity taxes are levied directly on industry (and are paid directly to government by industry), which we denote by E, and that customs duties are charged on imports used by industry, which we denote by $\bar{M}$, where $E'=[E_1\ E_2\ E_3]$ and $\bar{M}'=[\bar{M}_1\ \bar{M}_2\ \bar{M}_3]$. Dividing each E_i and $\bar{M}_i$ by the appropriate industrial output, q_i, gives the tax per unit of output for each industry, that is $[e_1\ e_2\ e_3]$ and $[\bar{m}_1\ \bar{m}_2\ \bar{m}_3]$. Following the standard national accounting convention, all indirect taxes charged to industry are passed on through increased prices to final demand. In economic terms, this involves an assumption that the price elasticity of demand for the taxed commodities is zero, which is one of the two possible extremes, the other would be that it is infinite so that none of the indirect taxes were passed on to final demand but were all met from factor income (this latter assumption is made in national accounting in the case of direct taxes).

Given our convention, then the estimate of "p" after levying these commodity taxes is

$$p=A'p+w+m+e+\bar{m} \tag{2}$$

$$=[I-A']^{-1}(w+m)+[I-A']^{-1}(e+\bar{m})$$

$$=b+[I-A']^{-1}(e+\bar{m}) \tag{3}$$

and

$$b=p-[I-A']^{-1}(e+\bar{m}) \tag{4}$$

In order to avoid matrix inversion, the SNA recommends (and employs in Table 2.1) the convention that an approximate basic price, b^*, be used rather than the true basic price. It nevertheless stresses the importance of using basic prices to achieve uniform valuation in an input-output table (see paras 2.9 and 2.10 for example and also Stone (1970), p. 173). One might also add that the use of basic prices is essential if one wishes to assess the inflationary effect of indirect taxes.

The approximate basic price, b^*, is estimated by:

$$b^*=A'p+w+m$$

and since, from (2),

$$p=b^*+e+\bar{m}$$

Then

$$\begin{aligned} b^* &= A'b^* + A'(e+\bar{m}) + (w+m) \\ &= [I-A']^{-1}A'(e+\bar{m}) + [I-A']^{-1}(w+m) \\ &= [I-A']^{-1}A'(e+\bar{m}) + b \end{aligned}$$

and we have a measure of the extent to which b^* exceeds b.

Returning to our numerical example, assume that the government now levies the following excise duties on industries: $E_1 = 10$, $E_2 = 30$ and $E_3 = 24$ and that import duties on imported inputs used by industry are $\bar{M}_1 = 5$, $\bar{M}_2 = 10$, $\bar{M}_3 = 6$. These taxes per unit of output (each tax is divided by the appropriate q_i in Table 1) are $e_1 = 0.1$, $e_2 = 0.15$, $e_3 = 0.08$ and $\bar{m}_1 = 0.05$, $\bar{m}_2 = 0.05$, $\bar{m}_3 = 0.02$.

Then

$$p_{(e+\bar{m})} = p_0 + [I-A']^{-1}(\bar{m}+e) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.308 \\ 0.314 \\ 0.215 \end{bmatrix} \begin{matrix} A(1) \\ B(2) \\ C(3) \end{matrix}$$

$$p_e = p_0 + [I-A']^{-1}e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.218 \\ 0.232 \\ 0.165 \end{bmatrix} \begin{matrix} A(1) \\ B(2) \\ C(3) \end{matrix}$$

$$p_{\bar{m}} = p_0 + [I-A']^{-1}\bar{m} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.090 \\ 0.082 \\ 0.050 \end{bmatrix} \begin{matrix} A(1) \\ B(2) \\ C(3) \end{matrix}$$

where $p_{(e+\bar{m})}$ is the producers' prices with both taxes levied, p_e the producers' prices with only excise duties levied, $p_{\bar{m}}$ the producers' prices if only customs duties are levied and p_0 the initial price, of unity, before taxes are levied.

Thus, the price increase in commodity $A(1)$ as a result of the taxes is 31 percent, of which 22 percent arose from the excise duty and 9 percent from the customs duty. The price rise was more than double the combined rate of tax imposed of 15 percent. In the case of commodity $B(2)$ the price increase is also 31 percent, which compares with a combined tax rate of 20 percent. For commodity $C(3)$ the price rise is 22 percent as compared to a combined tax rate of 10 percent.

If we had been using approximate basic prices instead of true basic prices, we would have calculated these as

$$b^* = p_{(e+\bar{m})} - (e+\bar{m})$$

or

$$\begin{bmatrix} 1.308 \\ 1.314 \\ 1.215 \end{bmatrix} - \begin{bmatrix} 0.15 \\ 0.20 \\ 0.10 \end{bmatrix} = \begin{bmatrix} 1.158 \\ 1.114 \\ 1.113 \end{bmatrix} \begin{matrix} A(1) \\ B(2) \\ C(3) \end{matrix}$$

The approximate basic price overstates the true basic price by 16 percent in the case of commodity $A(1)$ and by 11 percent in the case of commodities $B(2)$ and $C(3)$. We do not achieve uniform valuation. Equally, the price increase estimated as a result of the combined taxes is 13 percent $[(1.308/1.158) \times 100]$ in the case of commodity $A(1)$, 18 percent in the case of commodity $B(2)$ and 9 percent in the case of commodity $C(3)$, each of which is a little less than the combined rate of tax imposed, a necessary consequence of the method of defining b^*. Besides greatly understating the magnitude of the price increases, they are also very inadequate as a guide to the relative effects between commodities; for $A(1)$ we have true to approximate price-raising effects of 31:13, for $B(2)$ 31:18 and for $C(3)$ 22:9. It is perfectly possible to devise an example where a given commodity is charged a lower rate of tax than another commodity but ends up with a higher price increase than the other commodity due to the price effect of the complete package of indirect taxes upon it. However, this could never be estimated or revealed if approximate basic prices were used.

The implications of this are clear and important. One major use of SAM's is to explore income distribution and to estimate the impact of different government measures upon income distribution. It is thus possible that governments may endeavour to assist lower income groups by imposing lower rates of indirect taxes upon the commodities that they tend to consume and higher rates on commodities that higher income groups tend to consume and yet end up inducing, in the extreme case, higher relative price rises for the low-income commodities than for the high-income commodities. The extreme case may well be unlikely, since it depends upon the nature of inter-industry relationships, but it is perfectly possible that the relative degree of favour given to low-income groups will be substantially less than comparison of the tax rates would suggest. The use of approximate basic prices would wrongly confirm that government was achieving, almost exactly, its objectives. It is also possible, of course, that the relative degree of discrimination against high-income groups would be greater than comparison of the tax rates would indicate.

The fact that we have used the extreme national accounts assumption that indirect taxes are passed on entirely to final demand in no way affects the above argument. If we care, by the use of regression techniques or whatever, to estimate the proportion that is actually passed on, and this is less than unity for some or all commodities, then we simply charge that proportion as tax in our calculations. The overall price-raising effects will be less, but so will those of using approximate basic prices and the arguments over the possible bias of approximate basic prices, and of the possible perverse effects, remain.

The generalization from the above to considering the price effect of any cost change in the production of a commodity is immediate. We simply replace the vector $(e+\bar{m})$ in equations (2) and (4) by a vector for the particular changes of interest—an increase in wage rates or import prices in certain industries, or a vector of the proportion of direct taxes which it is estimated are passed on to final demand, if we wish to depart from the extreme national accounts convention that none is passed on, or whatever. An implicit assumption in the Leontief-type input-output model assumed is that price changes do not induce substitution. This is a drawback but nevertheless the results are still likely to be valuable approximations to reality. Moreover, if one has reason to feel that substitution effects may be substantial in certain cases, and can estimate these, then the model can be adjusted to incorporate them. An interesting example of using a SAM to estimate price effects, with limited incorporation of substitution effects, is given by Barker (1968).

It should also be noted that whilst the economic model employed in SAM's assumes that all commodity taxes are passed on to final demand, and this is what happens when true basic prices are calculated, this is not the case when approximate basic prices are used but there is no economic rationale for the proportion that is estimated as passed on.

2. Other Implications of the above Procedures

The estimates of true basic prices given by equation (4) will only be correct if (a) the proportionate increase in price resulting from the imposition of the indirect taxes is the same for all buyers, (b) the coefficients of the input-output table are those applicable for deriving basic prices and (c), related to this, production and uses, as estimated in a SAM, are the flows relevant to the transmission of price changes in a given time period. These points will be considered further.

In the example given, all three conditions were assumed to hold. In the case of (a), if for example, use of own production by an industry was charged at a different rate to use of its product by other industries, whilst exports and home consumption of its product were charged at different rates again, then equation (4) would not give correct results and an alternative estimation procedure must be devised. This problem seems more likely to arise with indirect taxes than other forms of price changes such as an increase in wage rates in a given industry.

Again in our example, we started from a position where basic prices were known, and considered how prices changed when indirect taxes were imposed. The coefficients of our A matrix, estimated in Table 1, were therefore calculated at the basic prices, and given our assumptions on elasticities and substitution effects, the answers resulting are correct. As shown, however, the price changes induced by the taxes were not in a constant proportion to the magnitude of the taxes and consequently if one started from a position where the taxes had already been levied, then the coefficients of the A matrix, now at producer's prices, would not be the same as those calculated at basic prices and adjustment of these for the known differential rates of taxation over the commodities would still not produce an A matrix the same as that at basic prices.

Consequently, estimates of true basic prices and price-raising effects would be incorrect if we tried to estimate these retrospectively using equation (4) with an A matrix derived from data at producers prices. Thus we have to distinguish between the situation of forecasting what the effects of price changes will be from one of estimating what they have been. Equation (4) is correct for the former, but not the latter. It seems likely that an iterative solution could be used in the latter case by re-estimating our A matrix after each calculation, linking to our given and fixed initial producers' prices, until we obtain the same estimate of basic prices and price-raising effects whether working forwards or backwards. But this is a major additional complication to the procedure.

It has been convenient to date to think in terms of basic prices and producers prices, but this is not necessary. If we wish to forecast the effect of any price change affecting the costs of industry, the relevant price situation is that existing prior to the change, which is likely to be one in which a variety of indirect and direct taxes are already being levied. The choice of valuation for the initial A matrix would then be between existing producers' and purchasers' prices. Depending upon the way in which it is considered that industries pass on their cost increases and the extent to which production activities provide their own marketing and transportation services, it could well be that in certain cases purchasers' prices may be preferable to producers' prices, or even a mixture of the two. The prices required are those actually charged by producers since it is these that reflect the effects of changes in costs that the producers have experienced.

Finally, the relevance of condition (c) is that a SAM estimates production and uses in a given time period, whereas cost changes are passed on by industries through sales and purchases in the given time period. In periods of major price change and substantial stock change this would introduce errors in estimates of basic prices or price-raising effects.

3. An Alternative Method of Estimating Price Effects

The use of inverse matrices in the above calculations whilst being a very convenient way of tackling the problem given access to modern computers, is rather like having a magic box—numbers are fed in and answers come out, but it is not readily apparent what has happened inside the box to produce the desired result. Moreover, the answers are only correct in certain limited circumstances, as already explained.

The following approach still requires a computer for solution, because it involves a large number of calculations, but it does not necessarily involve matrix inversion. More important, it traces the effect of a given cost through the economy with a logic that should make what is happening intuitively clear and can provide correct solutions in certain circumstances where the first method cannot. The results can be expressed simply in matrix algebra, and are given in this form at the end of this section. Those who prefer matrix algebra may wish to skip immediately to this, but we find that for purposes of understanding the procedure, the following approach is illuminating, although not easy to follow because it is tracing an inherently complex system.

Consider again the case of three industries (each producing its own characteristic product only) and final demand with the following table of coefficients:

TABLE 2

		Industry			Final	
		1(*A*)	2(*B*)	3(*C*)	Demand	Total
Commodity	*A*(1)	p_{11}	p_{12}	p_{13}	p_{1m}	1
	B(2)	p_{21}	p_{22}	p_{23}	p_{2m}	1
	C(3)	p_{31}	p_{32}	p_{33}	p_{3m}	1

The p_{ji} represent the proportion of total production of commodity "j" that is used by industry "i". The p_{jm} represent the proportion of commodity "j" purchased by final demand and hence $\sum_i p_{ji} = 1$. Instead of deriving our coefficients using column (industry) totals as in the first section, we are now deriving them by using row (commodity) totals.

In Table 3 we have given an illustration of how these purchases might be regarded as flowing through the economy. For example, industry $1(A)$ uses $p_{21}=(1)_{21}$ of the total value of production of commodity $B(2)$, where $(1)_{21}$ is the value of the first round of transactions of industry $1(A)$ in commodity $B(2)$, $(2)_{21}$ the value of the second round, etc. This is contained in the final output of industry $1(A)$ of which p_{12} is bought by industry $2(B)$. This is entered under industry $2(B)$ commodity $B(2)$, in the first row of round (2) as $p_{12}(1)_{21}$. But industry $B(2)$ also uses its own product included in its use of the output of industry $3(C)$, namely $p_{32}(1)_{23}$ and in addition has further own consumption of its initial own consumption, namely $p_{22}(1)_{22}$. The sum of these, $(p_{12}(1)_{21}+p_{32}(1)_{23}+p_{22}(1)_{22})$ is denoted by $(2)_{22}$ under Industry $2(B)$, Commodity $B(2)$, round 2. The value of this component in industry $2(B)$'s total output is passed on to industries $1(A)$, $2(B)$, $3(C)$ and final demand in the proportions p_{21}, p_{22}, p_{23} and p_{2m} respectively, and so on. This process continues to infinity. The fact that each column in Table 3 has a finite sum can be readily proved.

The sum of the second column, for example, which is industry $(1)(A)$'s use of commodity $B(2)$ is:

$$\begin{aligned}
&(1)_{21}=p_{21}\\
&+(2)_{21}=p_{11}(1)_{21}+p_{21}(1)_{22}+p_{31}(1)_{23}\\
&+(3)_{21}=p_{11}(2)_{21}+p_{21}(2)_{22}+p_{31}(2)_{23}\\
&+\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&+(n)_{21}=p_{11}(n-1)_{21}+p_{21}(n-1)_{22}+p_{31}(n-1)_{23}\\
&=\overset{n}{S}_{21}=p_{11}\overset{(n-1)}{S_{21}}+p_{21}\overset{(n-1)}{S_{22}}+p_{31}\overset{(n-1)}{S_{23}}+p_{21}
\end{aligned}$$

In general, with $(m-1)$ industries, the sum for the ith industry's use of commodity "j" is:

$$\overset{n}{S}_{ji}=p_{ji}+\sum_{x=1}^{(m-1)} p_{xi}\overset{(n-1)}{S_{jx}}\ , \quad \begin{array}{l} i=1,\ldots,(m-1)\\ j=1,\ldots,(m-1)\end{array} \tag{9}$$

where p_{ji} is the direct use of commodity "j" by industry "i" and $\sum_{x=1}^{(m-1)} p_{xi}\overset{(n-1)}{S_{jx}}$ is its indirect use of commodity "j" included in industry "i"'s purchases of commodities $x=1,2,\ldots,(m-1)$. Here we are summing columns in Table 3.

As already mentioned, we have continued with our simplification that each industry only produces its characteristic product, so that the number of commodities is equal to the number of industries (both equal to $(m-1)$). Whilst this assumption is not necessary, the logic of the system considered is one whereby costs, such as commodity taxes, are charged to industries and are then passed on by them through sales of goods and services. Given this, then where an industry produces more than one commodity, it would normally be necessary to sub-divide its input structure according to each commodity produced in order to trace the effects of cost changes correctly. This would effectively mean that we had returned to a situation where the number of industries was equal to the number of commodities. If it was not necessary to sub-divide the input structure of an

TABLE 3

THE TRANSMISSION OF A TAX THROUGH INTER-INDUSTRY PURCHASES TO FINAL DEMAND

	Industry									Final Demand		
	1(*A*)			2(*B*)			3(*C*)					
	Commodity			Commodity			Commodity			Commodity		
Round	$A(1)$	$B(2)$	$C(3)$	$A(1)$	$B(2)$	$C(3)$	$A(1)$	$B(2)$	$C(3)$	$A(1)$	$B(2)$	$C(3)$
1	$p_{11}=(1)_{11}$	$p_{21}=(1)_{21}$	$p_{31}=(1)_{31}$	$p_{12}=(1)_{12}$	$p_{22}=(1)_{22}$	$p_{32}=(1)_{32}$	$p_{13}=(1)_{13}$	$p_{23}=(1)_{23}$	$p_{33}=(1)_{33}$	$p_{1m}=(1)_{1m}$	$p_{2m}=(1)_{2m}$	$p_{3m}=(1)_{3m}$
2	$\underbrace{p_{21}(1)_{12}+p_{31}(1)_{13}+p_{11}(1)_{11}}_{=(2)_{11}}$	$\underbrace{p_{21}(1)_{22}+p_{31}(1)_{23}+p_{11}(1)_{21}}_{=(2)_{21}}$	$\underbrace{p_{21}(1)_{32}+p_{31}(1)_{33}+p_{11}(1)_{31}}_{=(2)_{31}}$	$\underbrace{p_{12}(1)_{11}+p_{32}(1)_{13}+p_{22}(1)_{12}}_{=(2)_{12}}$	$\underbrace{p_{12}(1)_{21}+p_{32}(1)_{23}+p_{22}(1)_{22}}_{=(2)_{22}}$	$\underbrace{p_{12}(1)_{31}+p_{32}(1)_{33}+p_{23}(1)_{12}}_{=(2)_{32}}$	$\underbrace{p_{13}(1)_{11}+p_{23}(1)_{12}+p_{33}(1)_{13}}_{=(2)_{13}}$	$\underbrace{p_{13}(1)_{21}+p_{23}(1)_{22}+p_{33}(1)_{23}}_{=(2)_{23}}$	$\underbrace{p_{13}(1)_{31}+p_{23}(1)_{32}+p_{33}(1)_{33}}_{=(2)_{33}}$	$\underbrace{p_{1m}(1)_{11}+p_{2m}(1)_{12}+p_{3m}(1)_{13}}_{=(2)_{1m}}$	$\underbrace{p_{1m}(1)_{21}+p_{2m}(1)_{22}+p_{3m}(1)_{23}}_{=(2)_{2m}}$	$\underbrace{p_{1m}(1)_{31}+p_{2m}(1)_{32}+p_{3m}(1)_{33}}_{=(2)_{3m}}$
3	$\underbrace{p_{21}(2)_{12}+p_{31}(2)_{13}+p_{11}(2)_{11}}_{=(3)_{11}}$	$\underbrace{p_{21}(2)_{22}+p_{31}(2)_{23}+p_{11}(2)_{21}}_{=(3)_{21}}$	$\underbrace{p_{21}(2)_{32}+p_{31}(2)_{33}+p_{11}(2)_{31}}_{=(3)_{31}}$	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

industry producing more than one commodity in order to obtain correct results, then it would follow that the commodities could be treated as a single commodity, and again, we would have the number of commodities equal to the number of industries.

The commodity tax included in industry "i"'s purchase of good "j", c_{ji}, is:

$$(10) \qquad c_{ji} = t_j p_{ji} + p_{ji} \sum_{x=1}^{(m-1)} \overset{(n-1)}{S_{xj}} \; t_x = p_{ji}(t_j + \sum_{x=1}^{(m-1)} \overset{(n-1)}{S_{xj}} \; t_x)$$

and that included in purchases by final demand, which we continue to denote by the subscript "m", is

$$(11) \qquad c_{jm} = t_j p_{jm} + p_{jm} \sum_{x=1}^{(m-1)} \overset{(n-1)}{S_{xj}} \; t_x = p_{jm}(t_j + \sum_{x=1}^{(m-1)} \overset{(n-1)}{S_{xj}} \; t_x)$$

where the t_x, $x = 1, 2, \ldots, (m-1)$, are the commodity taxes charged in total to each industry "x". We can consider t_x as E_x or $\bar{M}_x$ or in total $(E_x + \bar{M}_x)$ where E_x and $\bar{M}_x$ are as defined in the first procedure. Equations (10) and (11) give the same answers in total as the first procedure. But they also give us additional information. The $\overset{(n-1)}{S_{xj}}$ represent the relative value of each commodity x_i included in (to take the case of equation (11)) commodity "j". Each of these has been taxed by its appropriate t_x, and these have been passed on to industry "j" in its purchases to produce commodity "j", and industry "j" has passed these on, as well as the commodity tax t_j levied directly on it, to final demand. In equations (10) and (11) we are summing corresponding rows for each round in Table 3, for industry "i" and final demand respectively. It turns out that the same information can be obtained from equation 3, a fact that we have not seen noted in the literature to date. Since we have followed the national accounts convention that all indirect taxes charged to industry are passed on to final demand, it follows that the sum of each column in final demand is unity, or when multiplied by the tax levied on commodity "j" by government, to the value of the tax, i.e.:

$$t_j(p_{jm} + \sum_{x=1}^{(m-1)} p_{xm} \overset{(n-1)}{S_{jx}} \; = t_j)$$

or

$$(12) \qquad p_{jm} + \sum_{x=1}^{(m-1)} p_{xm} \overset{(n-1)}{S_{jx}} \; = 1$$

This can be used to provide a correction factor when only a limited number of rounds have been calculated, for purposes of computing equations (10) or (11). An example of this is given in appendix (1), for interest rather than any other reason, because the programming and computation of equation (10) is so simple that there would be no point in calculating so few rounds that a correction factor was necessary, unless one was undertaking the calculations manually. Otherwise, one could use the equations which follow, using matrix inversion, and again the correction factor would not be required.

Up until this point, we have assumed that the same price is charged for a commodity regardless of purchaser. Given this, then the p_{ji} would be the same whether they were calculated from values at basic prices or at producers' prices and we would obtain the same estimate for the value of commodity taxes included in price whether we started from a position before or after the taxes had been levied, as is desired. This represents an important gain over equation (4) because there are probably many cost changes that are passed on proportionately, regardless of purchaser, such as wage increases.

The equations are, however, readily modified if different buyers are charged different rates of tax. Suppose the rate of tax applicable to industry "i"'s use of commodity "j" is r_{ji} and that the general rate of tax, to be defined, on commodity "j" is r_j. We now replace our requirement that the proportionate use of commodity "j" by industry and final demand should sum to unity, $\sum_i p_{ji} = 1$, by the requirement that the entire tax charged should be allocated to users in proportion to their use of the commodity and their rate of tax, namely:

$$\sum_{i=1}^{m} p_{ji}\frac{r_{ji}}{r_j} = 1$$

hence

(13) $$r_j = \sum_{i=1}^{m} p_{ji} r_{ji}$$

The generalization of equations (10) and (11) is then:

(14) $$c_{ji} = p_{ji} t_j \left(\frac{r_{ji}}{r_j}\right) + p_{ji} \sum_{x=1}^{(m-1)} \overset{(n-1)}{S_{xj}}\; t_x \left(\frac{r_{xj}}{r_x}\right)$$

where we can define "i" to apply to each industry and each sector of final demand. Equation (14) will give correct answers where we start from a given position and estimate what the effects of a change in indirect taxes will be. If we wish to estimate what the effects have been, however, then it would not be correct to calculate the p_{ji} from the value of production after the change has occurred. The values of the p_{ji} would have to reflect the way in which it was considered that the commodity taxes were passed on, so that they would be the same as those which would have been obtained had they been calculated from values which would have prevailed in the absence of the taxes in question. Where the only difference in price arises from the difference in tax rate, then the p_{ji} could be calculated from the volume of production or an index of this. Alternatively, an iterative solution could probably be devised.

The above procedures can readily be described in terms of matrix algebra. It will be recognized that the procedure given in Table 3 is a method of estimating a matrix inverse using the relationship:

(15) $$[I-P]^{-1} = I + P + P^2 + \dots$$

which has a finite sum provided the condition $0 \leq p_{ij} < 1$ is met.

If we denote corresponding industry and commodity totals in Table 1 by α, then the relationship between the p_{ij} coefficients calculated from commodity totals in Table 2 and the a_{ij} coefficients calculated from industry totals is:

$$a_{ij}\frac{\alpha_j}{\alpha_i} = p_{ij}$$

When dealing with the transposes of the matrices, this relationship becomes

$$a'_{ij}\frac{\alpha_i}{\alpha_j}=p'_{ij}$$

where a'_{ij} is the coefficient in the ith row and jth column of the transpose of the A matrix, namely A', and similarly for p'_{ij}. Thus

$$[I-A']^{-1}=[I-\hat{\alpha}^{-1}P'\hat{\alpha}]^{-1}$$

where $\hat{\alpha}$ is the diagonal matrix formed from the vector of column/row totals of Table 1. Similarly

$$[I-\hat{\alpha}A'\hat{\alpha}^{-1}]^{-1}=[I-P']^{-1} \tag{16}$$

The $\overset{n}{S}_{ji}$ of equation (9) are estimating the sum $(P+P^2+\ldots)$ in (15) above and, denoting the matrix of these coefficients by S we have

$$[I+S']=[I-P']^{-1}$$

In terms of matrix algebra, equation (10) can be written as:

$$c_{\cdot i}=\hat{p}_{\cdot i}[I-P']^{-1}t \tag{17}$$

which gives the commodity taxes included in industry "i"'s purchase of each commodity as elements of the vector $c'_{\cdot i}=[c_{1i}c_{2i}\ldots c_{(m-1)i}]$. Industry "$i$" here can also be defined to include final demand, sector m, which can be further subdivided as required. The relationship between (17) above and equation (3) can then be formed by removing the basic price, b, from equation (3) and converting the remainder into absolute terms, whereupon we have:

$$c_{\cdot i}=\hat{p}_{\cdot i}[I-P']^{-1}t=\hat{u}_{\cdot i}[I-A']^{-1}e$$

where $\hat{u}_{\cdot i}$ is the diagonal matrix formed from the vector of absolute values of the u_{ij} in Table (1), $u'_{\cdot i}=[u_{1i}u_{2i}\ldots u_{(m-1)i}]$ and we have used the relationships $p_{ji}=u_{ji}/\alpha_j$ and $e_i=t_i/\alpha_i$.

Equation (14) can be written as

$$c_{\cdot i}=\hat{p}_{\cdot i}[t_r+S'_r t]$$

where $t_r=[t_1r_{1i}/r_1\ t_2r_{2i}/r_2\ldots t_{(m-1)}r_{(m-1)i}/r_{(m-1)}]$ and S'_r is the matrix:

$$\begin{bmatrix} S_{11}r_{11}/r_1 & S_{21}r_{21}/r_2 & \ldots & S_{(m-1)1}r_{(m-1)1}/r_{(m-1)} \\ S_{12}r_{12}/r_1 & S_{22}r_{22}/r_2 & \ldots & S_{(m-1)2}r_{(m-1)2}/r_{(m-1)} \\ \vdots & \vdots & & \vdots \\ S_{1(m-1)}r_{1(m-1)}/r_1 & S_{2(m-1)}r_{2(m-1)}/r_2 & \ldots & S_{(m-1)(m-1)}r_{(m-1)(m-1)}/r_{(m-1)} \end{bmatrix}$$

or, equivalently S'_r is $\{[I-P']^{-1}-I\}$ with each element, S_{ij} multiplied by r_{ij}/r_i, the relative rate of tax charged to industry "j" for its purchases of commodities "i" $=1,2,\ldots,(m-1)$.

In effect, one has solved the dual of a conventional input-output tableau through the above procedures.

4. A NEW APROXIMATION

Arising from the logic of the above estimating procedure, one is led to enquire whether the $\overset{n-1}{S_{xj}}$ of equation (10) might be reasonably approximated by the p_{xj} of Table 2. Using the data in Table 1, then Table 2 becomes:

		Industry 1(A)	2(B)	3(C)	Final Demand	Total
Commodity	A(1)	0.1	0.5	0.3	0.1	1
	B(2)	0:1	0.05	0.3	0.55	1
	C(3)	0.1	0.0667	0.1	0.7333	1

Insofar as the p_{xj} can only be approximations to the $\overset{n-1}{S_{xj}}$ and are bound to understate the true value thereof, the approximation will be improved if we adjust them by using the identity, equation 12:

$$p_{jm} + \sum_{x=1}^{(m-1)} p_{xm} \overset{(n-1)}{S_{jx}} = 1$$

and calculate

$$p_{jm} + \sum_{x=1}^{(m-1)} p_{xm} p_{jx} = Z$$

then estimate $\overset{n-1}{S_{jx}}$ by

$$\widehat{\overset{n-1}{S_{jx}}} = p_{jx}\left(\frac{1-p_{jm}}{Z-p_{jm}}\right)$$

For example

$$\widehat{\overset{n-1}{S_{21}}} = 0.1 \times \frac{1-0.55}{0.807-0.55} = 0.1748$$

$$\widehat{\overset{n-1}{S_{22}}} = 0.05 \times \frac{1-0.55}{0.807-0.55} = 0.0874$$

Continuing with this for all cases gives the following estimated values, $\widehat{\overset{n-1}{S_{jx}}}$

Estimated Values of $\overset{n-1}{S_{jx}}$

	$\widehat{\overset{n-1}{S_{\cdot 1}}}$	$\widehat{\overset{n-1}{S_{\cdot 2}}}$	$\widehat{\overset{n-1}{S_{\cdot 3}}}$
$\widehat{\overset{n-1}{S_{1\cdot}}}$	0.1783	0.8913	0.5348
$\widehat{\overset{n-1}{S_{2\cdot}}}$	0.1748	0.0874	0.5245
$\widehat{\overset{n-1}{S_{3\cdot}}}$	0.2222	0.1482	0.2222

The commodity taxes included in the value of production can then be estimated by writing equation (10) as

$$\hat{c}_{ji} = p_{ji}\left(t_j + \sum_{x=1}^{(m-1)} \widehat{S_{xj}}^{n-1}\, t_x\right)$$

where $\hat{c}_{ji}$ stands for the estimated commodity tax included in industry "i"'s use of good "j". The approximation could equally be employed in equation (14).

For example, the commodity tax included in industry 1(A)'s use of commodity B(2) is

$$\hat{c}_{21} = p_{21}(t_2 + \widehat{S_{12}}^{n-1}\, t_1 + \widehat{S_{22}}^{n-1}\, t_2 + \widehat{S_{32}}^{n-1}\, t_3)$$

Using the values of our earlier example for excise duties, $t_1 = 10$, $t_2 = 30$ and $t_3 = 24$, therefore:

$$\begin{aligned}\hat{c}_{21} &= 0.1[30 + 0.8913 \times 10 + 0.0874 \times 30 + 0.1482 \times 24] \\ &= 0.1 \times 45.1 \\ &= 4.5\end{aligned}$$

Similarly

$$\begin{aligned}\hat{c}_{22} &= 0.05 \times 45.1 = 2.3 \\ \hat{c}_{23} &= 0.3 \times 45.1 = 13.5 \\ \hat{c}_{2m} &= 0.55 \times 45.1 = 24.8\end{aligned}$$

The correct answers can be obtained from the results derived earlier for our example using p_e, where the basic price of commodity A(1) was increased by 21.8 percent, that of B(2) by 23.2 percent and that of C(3) by 16.5 percent. Applying these percentages to the values in Table (1), then we obtain the following comparison of true and approximate measures:

Commodity taxes included in the value of production

		Industry						Final Demand	
		1(A)		2(B)		3(C)			
		True	Approx.	True	Approx.	True	Approx.	True	Approx.
Commodity	A(1)	2.2	2.2	10.9	11.2	6.5	6.7	2.2	2.2
	B(2)	4.7	4.5	2.3	2.3	13.9	13.5	25.6	24.8
	C(3)	4.9	5.0	3.3	3.4	4.9	5.0	36.2	37.0
	Total	11.8	11.7	16.5	16.9	25.3	25.2	64.0	64.0

As required, the total indirect taxes charged of $(10+30+24) = 64$ have been passed on to final demand. The approximate allocation of these is very good. Using equation (10), of course, gives the same "true" results.

As with the exact alternative estimator, the alternative approximation also estimates the contribution of each indirect tax to the final commodity tax passed on in the price of any given commodity. Thus, $\hat{c}_{21}$, the commodity tax included in industry $1(A)$'s purchase of commodity $B(2)$, which is estimated as 4.5, is built up in the following way.

	Approx.	True
Arising directly from the tax on $B(2) = (0.1 \times 30)$	=3.0	3.0
Arising indirectly from the tax on $B(2) = (0.1 \times 0.0874 \times 30)$	=0.26	0.54
Total from the tax on $B(2)$	=3.26	3.54
Arising indirectly from the tax on $A(1) = 0.1 \times 0.8913 \times 10$	=0.89	0.71
Arising indirectly from the tax on $C(3) = 0.1 \times 0.1482 \times 24$	=0.36	0.40
Total commodity tax $= (3.26 + 0.89 + 0.36)$	=4.5	4.7

The relative shares of the different commodity taxes passed on in the price of commodity $B(2)$ are of course constant, whoever the buyer, given the assumption that there is only one price for the commodity. The numbers under "True" above are the correct estimates of the composition of the true total of 4.7 using equation (10). Not surprisingly, the relative error here in the approximation is somewhat greater than overall.

CONCLUSION

Social Accounting Matrices and the input-output tables embedded in them have normally been used to estimate "real" changes in an economy, such as the effects of a change in final demand for the product of a particular industry. They are equally valuable, however, in estimating the effects of price changes, such as an increase in indirect taxes, or in wage rates in a given industry or industries, etc. The use of the standard formula for calculating approximate basic prices is not suitable for this, however, and could prove badly misleading. A new approximate formula has been proposed which does seem to be quite reasonable. Nevertheless, we would prefer that an exact formula should be employed.

Finally, we feel that current terminology used is misleading. At present, satisfactory distinction is not made between indirect taxes, of which commodity taxes are a sub-group, and the inflationary effect of such taxes, as estimated from basic prices. We suggest that the term "indirect taxes" should be used to describe the taxes actually paid by industries (including distributors) to Government, and which appear as charges in the accounts of industries. The price-raising effect of these taxes is not separately identified in the accounts of industries, but is included in the prices that they charge for their production and comprise the value of the tax on the particular industry in question and additional price rises induced by passsing on indirect taxes through inter-industry purchases. These can only be estimated by techniques such as those covered above, and we suggest that they be called "price effects".

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APPENDIX 1

The calculation of the $\overset{n}{S}_{ji}$ in equation (9), using the data of Table 1 transformed into coefficients as in Table 2, was made by computer. The calculations were made for rounds of $n = 4$, 8, 12 and 16. The results after four rounds and sixteen rounds were:

	$\overset{4}{S}_{\cdot 1}$	$\overset{4}{S}_{\cdot 2}$	$\overset{4}{S}_{\cdot 3}$
$\overset{4}{S}_{1\cdot}$	0.247477	0.686809	0.625131
$\overset{4}{S}_{2\cdot}$	0.173564	0.167672	0.437142
$\overset{4}{S}_{3\cdot}$	0.148039	0.157529	0.205716

	$\overset{16}{S}_{\cdot 1}$	$\overset{16}{S}_{\cdot 2}$	$\overset{16}{S}_{\cdot 3}$
$\overset{16}{S}_{1\cdot}$	0.263240	0.711067	0.658100
$\overset{16}{S}_{2\cdot}$	0.181545	0.180047	0.453863
$\overset{16}{S}_{3\cdot}$	0.153814	0.166462	0.217869

By round 16, the answers are already correct to four decimal places. They were already correct to three decimal places by round 12.

If, for some reason, we had wished to stop our calculations after four rounds only, and apply a correction factor, then we would derive the correction factor from equation (12), which requires:

$$p_{1m}\overset{n-1}{S}_{11} + p_{2m}\overset{n-1}{S}_{12} + p_{3m}\overset{n-1}{S}_{13} = 1 - p_{1m}$$

etc. This results in a correction factor for the row 1 estimates, $\overset{4}{S}_{1\cdot}$ of 1.0454, for row 2, $\overset{4}{S}_{2\cdot}$ of 1.0462, and for row 3, $\overset{4}{S}_{3\cdot}$ of 1.0569. Correcting the above round 4 results by these factors gives the following estimates of $\overset{\frown}{\overset{n-1}{S_{jx}}}$:

	$\overset{\frown}{\overset{n-1}{S_{\cdot 1}}}$	$\overset{\frown}{\overset{n-1}{S_{\cdot 2}}}$	$\overset{\frown}{\overset{n-1}{S_{\cdot 3}}}$
$\overset{\frown}{\overset{n-1}{S_{1\cdot}}}$	0.259	0.718	0.653
$\overset{\frown}{\overset{n-1}{S_{2\cdot}}}$	0.182	0.175	0.457
$\overset{\frown}{\overset{n-1}{S_{3\cdot}}}$	0.157	0.167	0.217

which may be compared with the results for sixteen rounds. If one had to perform the calculations manually, then bearing in mind such considerations as the quality of data that one is using, the adjusted results after four rounds might well be considered adequate in the example used.

Given access to a computer, however, then one might still wish to employ the iterative solution if it was desired to introduce constraints, the program being instructed to stop once the constrained value of a particular variable had been reached. In such a case, of course, no correction factor would be required.