EFFECT OF GROUPING OF DATA ON THE DIVERGENCE BETWEEN LASPEYRES AND PAASCHE FORMS OF QUANTUM INDEXES*

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Historical data for the manufacturing industry in the United States and United Kingdom are quoted, showing in most cases divergences between Laspeyres and Paasche forms of the quantum index that are by no means negligible. When the Paasche index for two of the series is recalculated with quantity indicators for industries instead of for products, the divergence is greatly reduced, and when quantity indicators for industry groups are substituted it almost disappears. This raises some questions about the practices of econometricians and statisticians, which are discussed. In a mathematical appendix by E. R. Coleman it is suggested that the grouping effect referred to does not depend on the particular way in which the data are grouped in most quantum indexes.

In a recent paper by Alterman and Marimont,[1] price indexes derived from U.S. manufacturing census results showed rather wide divergences between Laspeyres and Paasche forms. In the price changes for the periods 1947–1954 and 1954–1958 implied by the production indexes for manufacturing published by the Bureau of the Census and the Board of Governors of the Federal Reserve System, the average annual rates of increase were 2·6 per cent for both periods with earlier-year weights, and only 2·0 per cent or less with later-year weights.[2]

Divergences of this order are in accordance with others that the writer has observed in quantum indexes based on data from censuses of production.

These results are consistent with what might be expected from both index-number theory and other empirical studies of price and production indexes. Divergence between Laspeyres and Paasche indexes occurs if price-changes and quantity-changes have shown some degree of correlation, and it is evidently normal for them to do so. Because the Laspeyres index exceeds the Paasche index in all the above cases, the correlation must have been negative, and this again is normal.

However, the results do not support what appears to be a common view of constant-price measures in recent years: that the divergences tend to be negligible, or—at least by implication—not great enough to prevent results based on one form of index-number alone from being used, for example, in econometric work.

A possible explanation of this inconsistency is that "rebasing tests" of the constant-price series used in measuring growth rates, for example, are often

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[1]Prices and Price Analysis in the Framework of the National Accounts, *Review of Income and Wealth*, Series 16, No. 2, June 1970.

[2]*Ibid.*, p. 158.

TABLE 1

LASPEYRES–PAASCHE DIVERGENCES IN QUANTUM INDEXES

	United States						United Kingdom	
	1909–1937[a]	1939–1947[b]	1947–1954[c]	1947–1957[d]	1954–1958[e]	1958–1963[e]	1924–1948[a]	1948–1954[f]
Level of detail used for rebasing	38 industries	1700 product items (p. 8)	6000 product items (p. 14)	207 series (p. 30)	About 1100 product classes, about 10,000 separately reported products (p. 1)		28 industries	2400 product items
Indexes for final year (first year = 100)								
Laspeyres	382	184	131	155	111·2	133·7	181	132·0
Paasche	209	169	126	148	108·3	132·4	138	118·7
Rates of increase per annum: %								
Laspeyres	4·90	7·92	3·93	4·48	2·69	5·98	2·50	4·74
Paasche	2·67	6·78	3·36	4·00	2·01	5·77	1·35	2·90
Difference	2·23	1·14	0·57	0·48	0·68	0·21	1·15	1·84

[a] W. E. G. Salter, *Productivity and Technical Change* (Cambridge 1966), pp. 170, 152.
[b] *United States Census of Manufactures 1947* (Washington 1952), pp. 4, 18.
[c] *United States Census of Manufactures 1954*, Vol. IV (Washington 1958), pp. 4, 23.
[d] Federal Reserve Board, *Industrial Production Index, 1959 Revision* (Washington 1960), p. 31.
[e] *United States Census of Manufactures 1963*, Vol. IV (Washington 1968), p. 26. Indexes shown above for 1954–1958 are reciprocals of the indexes as published, which are 1958 = 100.
[f] Nicholson, R. J. and Gupta, S., Output and productivity changes in British manufacturing industry, *Journal of the Royal Statistical Society*, Series A, Vol. 123, Pt. 4 (1960), p. 438.

done by use of highly aggregated data—probably because in most cases disaggregated data are not available. With the exception of the two series quoted from Salter, all of the above series are based on highly disaggregated data.

In two of the examples based on disaggregated data, it is possible to demonstrate the effect of progressive aggregation of the data on the divergence between the two forms of the indexes. One is the United Kingdom data for 1948–1954, analysed in a paper by Nicholson and Gupta, who as far as I know were the first to draw special attention to the dependence of the Laspeyres–Paasche divergence on the level of disaggregation.[3] Nicholson and Gupta recalculated the 1948–1954 manufacturing production index using price weights for 1954 instead of 1948 for all products in the index, and then did the same again after replacing the original q's representing individual products by Laspeyres quantum indexes: first, indexes for 138 industries and then indexes for 14 industry groups. The other example is the United States data for 1947–54 for which the census volume

[3] Although Vol. IV of the *U.S. Census of Manufactures* for 1954 had pointed out that the divergence for the index for individual products was greater than that for the index for industries (p. 23).

provides separate indexes for 6,000 product items and 436 industries, and gives data which enable a similar index to be made for 21 industry groups. The rebasing of the product and industry indexes is done in the census volume itself; the rebasing of the industry group index was done by the writer, from data in Table D, p. 23.

The results of these analyses are shown in the following table:

TABLE 2

EFFECT OF GROUPING ON LASPEYRES–PAASCHE DIVERGENCES

	United States 1947–1954 (*1954 Census of Manufactures, Indexes of Production*, p. 23)			United Kingdom 1948–1954 (Nicholson and Gupta, p. 438)		
Level of detail used for rebasing	6000 product items	436 industries	21 industry groups	2400 product items	138 industries	14 industry groups
Indexes for final year (first year = 100)						
Laspeyres	131	131	130·9[a]	132·0	132·0	132·0
Paasche	126	129	130·5[b]	118·7	129·3	131·9
Rates of increase per annum (%)						
Laspeyres	3·93	3·93	3·92	4·74	4·74	4·74
Paasche	3·36	3·70	3·87	2·90	4·38	4·73
Difference	0·57	0·23	0·05	1·84	0·36	0·01

[a]Table D, p. 23, sum of the products obtained by multiplying the indexes in the third column by the proportions in the sixth column.

[b]Table D, p. 23, sum of the products obtained by multiplying the indexes in the third column by the proportions in the fifth column.

Evidently the negative correlation between quantity changes and price changes which has caused the divergence of Laspeyres and Paasche indexes is much less marked, in these two examples, for grouped than for ungrouped data. For the "industries," measured in hundreds, the divergence is much narrower than that shown for the product items, measured in thousands, and for the industry groups, measured in tens, it has almost disappeared.

Does the effect depend upon any special economic properties of the groupings adopted? I have the impression that it does not—in other words, that the effect is due to the "averaging" effect of grouping, and would tend to occur in some degree whatever the principle on which the group indexes had been formed. The mathematical appendix by a colleague, E. R. Coleman, tends to confirm this impression.

It would be interesting to extend this kind of empirical testing, if data could be found. Some of the empirical data quoted earlier are inconclusive. Some of the wider divergences in the examples in Table 1 occurred when the rebasing was done for broad industry groups, relatively few in number. Does this mean that the divergence would have been greater still had it been done for finer industry groups or individual products? Further testing of data would help to elucidate any differences which may exist between different types of case. The kind of data needed, that is, data on quantities and prices (or unit values) for two dates, in great detail and in grouped form, preferably at two levels at least, is likely to be available only from factory censuses and household expenditure surveys.[4]

Meanwhile certain questions arising from the above results seem to be worth asking. If the Laspeyres–Paasche divergence may be much greater for ungrouped data than for grouped data, can we attach much significance to tests that have been based on grouped data because, as is most often the case, nothing else is available? This question is relevant to studies of alternative measures of relative growth rates in different countries, for example, those done by J. McGibbon for O.E.C.D. (*Productivity Measurement Review* No. 36, February 1964, O.E.C.D., Paris, pp. 9–12). McGibbon used 12 major expenditure groups as the detail for rebasing, and recalculated the GNP aggregates for 11 countries at prices of the beginning and the end of the period; in all cases the effect on the growth rate was negligible. McGibbon was aware of the effects of aggregation on the size of the divergence: "The full effect of the selection of the base year for constant price series can only be seen in those countries which have made a detailed recalculation." (p. 10). He quotes results for six such countries, which still show fairly narrow divergences, but does not state at what level of aggregation the detailed rebasing was done.

This is allied to the question raised earlier: whether the use of quantum indexes untested for base-change, which is common in much econometric work, would be so common if it were not believed that such tests as have been carried out on these types of data (with grouped data) usually revealed only narrow divergences.

A further question concerns the practice, which must be fairly common, of measuring flows of goods and services at constant prices by dividing their value by a Laspeyres price index. This is generally seen to provide only an

[4]Household expenditure data were used recently by R. F. Fowler to test the divergences between Laspeyres and Paasche versions of retail price and consumption quantum indexes for the United Kingdom 1958–1967. (R. F. Fowler, *Some problems of index-number construction*, Studies in Official Statistics, Research Series No. 3, Central Statistical Office, London 1970; summarised in *Employment and Productivity Gazette*, March 1970). The results were:

	Price index 1958–1967	Quantum index 1958–1967
Level of detail	92 sections	92 sections
Indexes for 1967 (1958 = 100)		
Laspeyres	129·45	124·73
Paasche	123·40	118·91
Rates of increase per annum (%)		
Laspeyres	2·91	2·49
Paasche	2·37	1·94
Difference	0·54	0·55

approximation to a Laspeyres quantum index, but it is claimed that the approximation will be closer than it otherwise would if the value series has first been separated into as many classes as possible, and each class divided by its own Laspeyres price index.[5] This is true, but the above examples suggest that in practice the result may not get very close to a true Laspeyres quantum index in this way. The limit is reached only when the classes whose values are being divided by Laspeyres price indexes become individual commodities. The empirical results appear to suggest that the really significant step is between classes and commodities, not between the global aggregate and classes. And in practice the data available usually relate only to fairly broad classes.

Additionally, if the degree of aggregation can make such a difference to the results of rebasing, should not statisticians take care to indicate the level of detail at which the rebasing was done by specifying the number of price or quantum series involved whenever they present such results?

Finally, in the light of the foregoing one might question some of the remarks in the revised SNA[6] on methods considered acceptable for expressing values at constant prices by the use of price indexes—especially paragraphs 4.45, 4.46 and 4.48.

Appendix

The Effect of Aggregation on the Divergence between Laspeyres' and Paasche's Index Numbers

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When both indexes are calculated at the same level of aggregation there is a divergence due to the different weighting systems; Laspeyres' index is usually larger than Paasche's.

Let us suppose we have J groups of commodities, the jth group containing n_j items, with N items altogether. Denoting base year quantities by p_{ij} etc., and current year by P_{ij} etc., Laspeyres' quantum index is

$$(1) \qquad I_L = \frac{\sum_{j=1}^{J} \sum_{i=1}^{n_j} Q_{ij} p_{ij}}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} q_{ij} p_{ij}}$$

[5]See, for example, the U.N. Statistical Commission paper *Estimates of Product and Expenditure at Constant Prices*, E/CN.3/322, paras. 27 and 36.

[6]United Nations Statistical Office, *A System of National Accounts*, Studies in Methods, Series F, No. 2, Rev. 3 (New York, 1968).

while the Paasche index is

$$(2) \qquad I_P = \frac{\sum_{j=1}^{J} \sum_{i=1}^{n_j} Q_{ij} p_{ij}}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} q_{ij} P_{ij}}.$$

The difference $I_P - I_L$ can be expressed in the following way.

Put

$$m_{ij} = p_{ij} q_{ij}$$
$$k_{ij} = Q_{ij}/q_{ij}$$
$$r_{ij} = P_{ij}/p_{ij}$$

Then

$$\begin{aligned}\Delta = I_P - I_L &= \Sigma kr \frac{m}{\Sigma rm} - \Sigma k \frac{m}{\Sigma m} \\ &= \frac{1}{\Sigma rm}\left[\Sigma krm - \frac{\Sigma mk \Sigma mr}{\Sigma m}\right] \\ &= \frac{1}{\Sigma rm}\left[\Sigma m\left(k - \frac{\Sigma mk}{\Sigma m}\right)\left(r - \frac{\Sigma mr}{\Sigma m}\right)\right] \\ &= \frac{1}{L_r} \cdot \frac{\Sigma m(k - L_k)(r - L_r)}{\Sigma m}.\end{aligned}$$

where L_k and L_r are Laspeyres' indexes of quantum and price respectively.

Now the *weighted coefficient of correlation* (between quantum- and price-relatives) is defined by

$$r_{m:k,r} = \frac{\Sigma m(k - L_k)(r - L_r)}{\sigma_{m:k}\sigma_{m:r}\Sigma m}$$

where

$$\sigma_{m:k} = \left(\frac{\Sigma mk^2}{\Sigma m}\right)^{1/2}, \qquad \sigma_{m:r} = \left(\frac{\Sigma mr^2}{\Sigma m}\right)^{1/2}$$

are weighted standard deviations.

(NOTE: all sums in the preceding are over all items.)

Therefore:

$$(3) \qquad \Delta = r_{m:k,r} \times \sigma_{m:k} \times \frac{\sigma_{m:r}}{L_r}.$$

So the *sign* of the divergence is determined by the sign of the weighted coefficient of correlation.

It can be shown by an analogous argument that the divergence for price indexes is exactly the same, except that L_k replaces L_r.[7]

Now we are interested in the case where these indexes are calculated from grouped data. (So that each k and r is itself an index.)

[7]The results so far are standard.

Looking at equation (3) for this case, it seems plausible that the divergence should be smaller, since

(a) L_r will be about the same (see below); it is exactly the same if group values are used as weights;

(b) the grouped weighted standard deviations are almost certain to be smaller than the ungrouped standard deviations (this follows from the normal relationship between population variance and sample mean variance, but the usual conclusion is complicated by the variable effect of the weighting; this in turn depends on correlation between the weights and the *squared* deviations);

(c) it follows from these remarks that the phenomenon will be observed unless the weighted correlation coefficient for the grouped data is sufficiently greater than the weighted correlation coefficient to offset the effect of (b). Since the standard deviations will each be greater by a factor of order $\sqrt{N/J}$ in the grouped case (assuming the effect of the weighting is not extreme), the phenomenon will not occur provided the weighted correlation coefficient for the groups exceeds the coefficient for the ungrouped data by a factor greater than N/J. The same reasoning suggests that the phenomenon becomes more marked as aggregation increases (i.e. as J decreases), at least up to the point where the standard deviations cannot be calculated (i.e. one must have at least two groups).

These somewhat heuristic conclusions are reinforced by the following alternative analysis. The case is the most obvious one, in which group quantum indexes are weighted with group price indexes. This is the natural procedure to use when *only* the group figures are available. There are of course alternative methods.

Let w_{ij} be the weight in the disaggregative index

$$\left(\frac{P_{ij}q_{ij}}{\sum_{j=1}^{J}\sum_{i=1}^{n_j} P_{ij}q_{ij}} \text{ for Paasche's index, etc.}\right)$$

v_{ij} be the weight in the aggregative index at the group level

$$\left(\text{i.e. } Q_j = \sum_{i=1}^{n_j} k_{ij}v_{ij}, \qquad v_{ij} = \frac{P_{ij}q_{ij}}{\sum_{i=1}^{n_j} P_{ij}q_{ij}} \text{ for Paasche's index}\right)$$

P_j be the jth group price index (of the appropriate form).

Then the difference between the aggregative (A) and disaggregative (D) index (of a particular type, e.g. Paasche's) is

$$A - D = \sum_{j=1}^{J}\left(\sum_{i=1}^{n_j} k_{ij}v_{ij}\right)\frac{P_j}{\sum_{j=1}^{J} P_j} - \sum_{j=1}^{J}\sum_{i=1}^{n_j} k_{ij}w_{ij}$$

$$= \sum_{j=1}^{J} \sum_{i=1}^{n_j} k_{ij} \left(v_{ij} \cdot \frac{P_j}{\sum_{j=1}^{J} P_j} - w_{ij} \right)$$

$$= \sum_{j=1}^{J} \sum_{i=1}^{n_j} k_{ij} \left(v_{ij} \frac{P_j}{\sum_{j=1}^{J} P_j} - \frac{v_{ij}}{J} \right) + \sum_{j=1}^{J} \sum_{i=1}^{n_j} k_{ij} \left(\frac{v_{ij}}{J} - w_{ij} \right)$$

$$= \sum_{j=1}^{J} \left(\frac{P_j}{\sum_{j=1}^{J} P_j} - \frac{1}{J} \right) \sum_{i=1}^{n_j} k_{ij} v_{ij} + \sum_{j=1}^{J} \sum_{i=1}^{n_j} k_{ij} \left(\frac{v_{ij}}{J} - w_{ij} \right)$$

(4)
$$= - \sum_{j=1}^{J} \sum_{i=1}^{n_j} k_{ij} \left(w_{ij} - \frac{v_{ij}}{J} \right) + \sum_{j=1}^{J} \left(\frac{P_j}{\sum_{j=1}^{J} P_j} - \frac{1}{J} \right) Q_j$$

Putting

$$d_{ij} = w_{ij} - \frac{v_{ij}}{J}, \qquad \delta_{ij} = \frac{P_j}{\sum_{j=1}^{J} P_j} - \frac{1}{J}$$

we find

(5)
$$A - D = -N r_{d,k} \cdot \sigma_d \cdot \sigma_k + J \cdot r_{\delta,Q} \cdot \sigma_\delta \cdot \sigma_Q$$

The implications of this result are as follows.

1. For Laspeyres' index,
   (a) $\sigma_\delta = 0$, since weighting is by base-year price indexes, all of which are 1 by definition. So the second term is zero;
   (b)
   $$d_{ij} = \frac{p_{ij} q_{ij}}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} p_{ij} q_{ij}} - \frac{p_{ij} q_{ij}}{J \sum_{i=1}^{n_j} p_{ij} q_{ij}}$$
   One expects $k_{ij} = Q_{ij}/q_{ij}$ to be negligibly correlated with this, since changes are generally assumed to be exogenous to the base year price and quantum structure.
   So the first term should be close to zero too.

This justifies the assertion made above (point (a) under the equation (3)).

It follows from these remarks that the change in divergence due to aggregation depends basically on the behaviour of the Paasche index when aggregative data is used.

2. For Paasche's index,
   (a) Note that due to the arrangement of terms in equation (4), the first term of equation (5) has effect opposite in sign to that of the correlation coefficient in it. Now the first term represents an effect due to the

correlation between price and quantum changes within groups, whereas the second is due to correlation between groups. However, as explained above, the standard deviation factors will be significantly smaller in the second term (each by order $\sqrt{N/J}$); J is by definition smaller than N, so the first term will predominate unless $r_{\delta,Q}/r_{d,k}$ is of order greater than N. So unless the grouped correlation coefficient is this much greater than the ungrouped coefficient, the aggregative index will be greater and the divergence less than for the disaggregative indexes. (This assumes negative correlation coefficients; if both are positive, the aggregative index will be smaller, but in this case Paasche's index will generally exceed Laspeyres', so that the divergence is again smaller than for disaggregative indexes. If $r_{\delta,Q} > 0$ and $r_{d,k} < 0$ the aggregative index will be greater whatever the relative sizes of $r_{d,k}$ and $r_{\delta,Q}$ and so once again there will be a smaller divergence. The final case, $r_{\delta,Q} < 0$ and $r_{d,k} > 0$ results in the opposite phenomenon, an increased divergence.)

(b) For a greater degree of aggregation, J is of course smaller, and the standard deviations σ_δ and σ_Q will also decrease.

So the second term will be smaller unless the coefficient of correlation increases sufficiently to offset those changes. If we take $\sigma_\delta \doteqdot \sigma_d/\sqrt{N/J}$ as suggested above, we have

$$\frac{Jr_{\delta,Q}(J)\sigma_\delta(J)\sigma_Q(J)}{J'r_{\delta,Q}(J')\sigma_\delta(J')\sigma_Q(J')} = \left(\frac{J}{J'}\right)^2 \cdot \frac{r(J)}{r(J')}$$

so that the correlation coefficient must increase at least as the inverse square of the number of groups, or else the effect of this term is to decrease the divergence $A-D$ as the level of aggregation increases.

However, for greater aggregation the term $w_{ij}-v_{ij}/J$, which expresses the difference in price-weighting given to the quantum-relative $k_{ij} = Q_{ij}/q_{ij}$ in the disaggregative and aggregative indexes, will be more highly correlated with k_{ij}. (The reason for this is as follows. We have

$$w_{ij} - \frac{v_{ij}}{J} = P_{ij}q_{ij}\left(\frac{1}{\sum_{j=1}^{J}\sum_{i=1}^{n_j} P_{ij}q_{ij}} - \frac{1}{J\sum_{i=1}^{n_j} P_{ij}q_{ij}}\right),$$

for a particular item, say the i_0th in the j_0th group,

$$= P_{i_0j_0}q_{i_0j_0}\left(\frac{1}{\sum_{\substack{j=1 \\ (i,j)\neq(i_0,j_0)}}^{J}\sum_{i=1}^{n_j} P_{ij}q_{ij}+P_{i_0j_0}q_{i_0j_0}} - \frac{1}{J\sum_{\substack{i\neq i_0 \\ i=1}}^{n_j} P_{ij}q_{ij}+JP_{i_0j_0}q_{i_0j_0}}\right)$$

For a greater degree of aggregation, J is smaller, and so the bracketed term is more stable—for a given P-change it will change less—hence the whole term more closely follows the behaviour of P_{ij}; therefore $r_{d,k}$ is larger.)

These remarks indicate that the divergence tends to decrease with increasing aggregation, unless this second effect dominates the first.

The possibility for this aggregative effect not to occur seems unlikely in economic circumstances where there are usually negative correlations between price and quantity changes. Even so, there would be no effect if the two terms of equation (5) were equal. From what has been said above, it seems that this would require an unusual relation between the two pairs of standard deviations or else a very high value for the ratio of the grouped correlation coefficient to the ungrouped correlation coefficient which would occur only if the groups were selected on an arbitrary principle such as grouping separately (or together) commodities with the same quantum relatives.

In the situation where there was no negative correlation between quantity and price changes *within* groups, but some correlation *between* groups, equation (5) indicates that the disaggregative index, D, would be bigger than the aggregative index A. A situation almost like this, but not so extreme, could produce a divergence of zero. That is, the within-group correlation would have to be very much smaller, if negative, than the between-group correlation.

There are alternatives for combining the group indexes; in general, the same kind of conclusion seems reasonable—a between-groups effect and a within-group effect; the direction of the divergence will depend on the relative sizes of the factors discussed above. (In the case where individual quantum relatives are replaced by group indexes in the Paasche formula, a similar kind of analysis produces two terms which both act to decrease the divergence as aggregation increases.)